

CARDINAL-PRESERVING EXTENSIONS

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Cardinal-Preserving Extensions

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Abstract A classic result of Baumgartner-Harrington-Kleinberg [1] implies that assuming CH a stationary subset of ω_1 has a CUB subset in a cardinal-preserving generic extension of V , via a forcing of cardinality ω_1 . Therefore, assuming that ω_2^L is countable: $\{X \in L \mid X \subseteq \omega_1^L \text{ and } X \text{ has a CUB subset in a cardinal-preserving extension of } L\}$ is constructible, as it equals the set of constructible subsets of ω_1^L which in L are stationary. Is there a similar such result for subsets of ω_2^L ? Building on work of M. Stanley [9], we show that there is not. We shall also consider a number of related problems, examining the extent to which they are “solvable” in the above sense, as well as defining a notion of reduction between them.

We assume throughout that $0^\#$ exists.

Definition A subset X of L is Σ_1^{CP} iff X can be written in the form

$a \in X$ iff $\varphi(a)$ holds in a cardinal-preserving extension of L

for some Σ_1 formula φ . We intend our cardinal-preserving extensions of L to satisfy AC and to be contained in a set-generic extension of V . (In all natural cases, the words “a set-generic extension of” can be omitted; see the Remark following the proof of Theorem 1 below.)

Theorem 1 *If X is Σ_1^{CP} then X is constructible from $0^\#$.*

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Theorem 2 $0^\#$ is Σ_1^{CP} . And there are Σ_1^{CP} sets of constructibility degree strictly between 0 and $0^\#$.

Theorem 3 The following Σ_1^{CP} sets are equiconstructible with $0^\#$:

(a) $\{T \mid T \in L \text{ and } T \text{ is a tree on } \kappa \text{ of height } \kappa \text{ with a cofinal branch in a cardinal-preserving extension of } L\}$, for κ an infinite successor cardinal of L .

(b) $\{X \subseteq \kappa \mid X \in L \text{ and } X \text{ contains a CUB subset in a cardinal-preserving extension of } L\}$, for κ regular in L , $\kappa > \omega_1^L$.

(c) $\{X \subseteq \kappa \mid X \in L \text{ and } X \text{ is the set of ordinals } < \kappa \text{ which are admissible relative to some real in a cardinal-preserving extension of } L\}$, for κ an uncountable cardinal of L .

(d) $\{X \subseteq \kappa \mid X \in L \text{ and } X \text{ is the intersection with } \kappa \text{ of a class which is } \Delta_1\text{-definable over } L[R] \text{ without parameters, for some real } R \text{ in a cardinal-preserving extension of } L\}$, where κ is an L -cardinal greater than ω_2^L .

Theorem 3 will be proved by “reducing” $0^\#$ to the sets mentioned. In fact we shall need the following more general notion of “reduction”.

Definition Suppose that (X_0, X_1) and (Y_0, Y_1) are pairs of disjoint subsets of L . Then we write

$$(X_0, X_1) \longrightarrow_L (Y_0, Y_1)$$

iff there is a function F in L such that

$$a \in X_0 \rightarrow F(a) \in Y_0$$

$$a \in X_1 \rightarrow F(a) \in Y_1.$$

We write X_0 instead of (X_0, X_1) in case X_0 is the complement (within some constructible set) of X_1 , and similarly for the Y 's. It is clear that if (X_0, X_1) is nonconstructible and $(X_0, X_1) \longrightarrow_L (Y_0, Y_1)$, then (Y_0, Y_1) is also nonconstructible. In the proof of Theorem 3 we shall obtain reductions in this sense of $0^\#$ to the sets mentioned.

Theorem 3 suggests that the Baumgartner-Harrington-Kleinberg result should be viewed as a rare example of a nontrivial “solvable” Σ_1^{CP} problem. However it is not the only such example:

Theorem 4 If κ is ω_2^L in the set described in Theorem 3 (d), then the resulting set is constructible.

Proof of Theorem 1 Suppose that X is Σ_1^{CP} and therefore can be written in the form

$$a \in X \text{ iff } \varphi(a) \text{ holds in a cardinal-preserving extension of } L$$

where φ is Σ_1 .

Suppose that a belongs to X and choose a set of ordinals S in a set-generic extension $V[G]$ of V such that $\varphi(a)$ is true in $L[S]$ where $L[S]$ is a cardinal-preserving extension of L . Choose α so that $X \subseteq L_\alpha$, $S \subseteq \alpha$ and the parameters in φ belong to L_α . Let c be a real generic over $V[G]$ for a constructible set-forcing such that α and the parameters in φ are coded by c . The set S is countable in $V[G][c]$. Thus

$$Y_a = \{b \in V[G][c] \mid b \text{ is a real and } b \text{ codes both a set of ordinals } S \text{ such that } L[S] \text{ is a cardinal-preserving extension of } L \text{ satisfying } \varphi(a), \text{ together with a witness for } \varphi(a)\}.$$

is nonempty. Moreover Y_a is Π_2^1 in c and has an element b constructible from the pair c, S . As c is set-generic over $V[G]$ and therefore over $L[S]$, the pair b, c does not construct $0^\#$. It follows from the main result of Harrington-Kechris [6] that Y_a has an element b in $L[c^\#] = L[0^\#, c]$, and the latter is a set-generic extension of $L[0^\#]$. As this applies to any element a of X , we have:

$a \in X$ implies

$\varphi(a)$ is true in a cardinal-preserving extension of L included in a set-generic extension of $L[0^\#]$.

Conversely, if $\varphi(a)$ is true in a cardinal-preserving extension of L included in a P -generic extension $L[0^\#]$ for some forcing $P \in L[0^\#]$, then this fact is forced by some condition $p \in P$ and therefore $\varphi(a)$ is true in a cardinal-preserving extension of L included in a P -generic extension of V . Thus the above implication is in fact an equivalence, giving $X \in L[0^\#]$. \square

Remark Suppose that X is Σ_1^{CP} via the formula φ , X is a subset of L_α for some countable ordinal α and the parameters in φ are countable. Then the previous proof shows that if a belongs to X , then $\varphi(a)$ has a solution in a cardinal-preserving extension of L contained in V . This is because in V

there exist generics for any countable constructible forcing. It follows that in this special case, we can equivalently define Σ_1^{CP} using cardinal-preserving extensions of L which exist in V , and not only in a set-generic extension of V .

Proof of Theorem 2 Notice that any set of the form

$$n \in X \text{ iff } \exists R(\varphi(n, R) \wedge \text{Card}^{L[R]} = \text{Card}^L),$$

where φ is Π_2^1 without parameters, is Σ_1^{CP} , as for reals R that preserve L -cardinals, $\varphi(n, R)$ holds exactly if it holds in $L_{\omega_1^L}[R]$. So to show that $0^\#$ is Σ_1^{CP} it suffices to show that it can be written in the above form.

In [3] a notion of “guess” at a finite sequence of Silver indiscernibles was defined, together with a notion of “killing a guess”. It was shown that actual finite sequences of Silver indiscernibles cannot be killed, but there are reals that preserve cardinals over L which kill all guesses which “explicitly contradict” $0^\#$. Using these methods one can show:

If n does belong to $0^\#$ then there is a real R which preserves cardinals over L such that R kills every guess which says that n does not belong to $0^\#$.

As finite sequences of actual Silver indiscernibles cannot be killed, it follows that $n \in 0^\#$ iff there is a real R which preserves cardinals over L such that R kills every guess which says that n does not belong to $0^\#$, and therefore $0^\#$ is Σ_1^{CP} .

In [3] it was also shown that there is a real R such that:

- (1) R is nonconstructible and preserves L -cardinals.
- (2) $n \in R$ iff R kills every guess which says that n does not belong to R iff there exists a real S such that S preserves L -cardinals and S kills every guess which says that n does not belong to S .

It follows that R is Σ_1^{CP} , and by (1), R has L -degree strictly between 0 and $0^\#$. \square

Proof of Theorem 3 (a) For κ regular in L we let $\mathcal{T}(\kappa)$ denote $\{T \mid T \in L \text{ and } T \text{ is a tree on } \kappa \text{ of height } \kappa \text{ with a cofinal, cardinal-preserving branch}\}$,

where in general a set a is *cardinal-preserving* if L and $L[a]$ have the same cardinals. It is sufficient to show that $0^\# \rightarrow_L \mathcal{T}(\kappa)$, when κ is the L -successor to an infinite L -cardinal λ . For this purpose we again use the “guess-killing” method of [3]. Let T_n consist of all $s \in L, s : |s| \rightarrow 2, \lambda \leq |s| < \kappa$ such that:

(*) For all $\lambda < \eta \leq |s|$, if $L_\beta[s \upharpoonright \eta] \models ZF^- + \eta = \lambda^+$ then $L_\beta[s \upharpoonright \eta] \models$ Every guess which says that n does not belong to $0^\#$ has been killed.

Using the methods of [3] it can be shown that T_n has a cofinal cardinal-preserving branch when n belongs to $0^\#$. But note that by reflection, if T_n has a cardinal-preserving branch b then in $L[b]$ every guess which says that n does not belong to $0^\#$ has been killed, and therefore n really does belong to $0^\#$. It follows that $n \in 0^\#$ iff T_n belongs to $\mathcal{T}(\kappa)$, as desired.

(b) We first improve the result of part (a). Let κ be the L -successor to a regular L -cardinal λ . Define:

$$\begin{aligned} \mathcal{T}^*(\kappa) &= \{T \in \mathcal{T}(\kappa) \mid T \text{ is } \Delta_1\text{-definable over } L_\kappa \text{ from the parameter } \lambda \text{ and } T \\ &\text{has a cardinal-preserving, stationary, } \mathcal{P}(\lambda)\text{-preserving cofinal branch}\} \\ \mathcal{T}^{**}(\kappa) &= \{T \in \mathcal{T}(\kappa) \mid T \text{ is } \Delta_1\text{-definable over } L_\kappa \text{ from the parameter } \lambda \text{ and } \\ &T \text{ has a cardinal-preserving cofinal branch}\}, \end{aligned}$$

where $b \subseteq \kappa$ is *stationary* if in $L[b]$ its intersection with $(\text{cof } \lambda)^L$ is stationary, and b is $\mathcal{P}(\lambda)$ -*preserving* if L and $L[b]$ have the same subsets of λ . We first show: $0^\# \rightarrow_L (\mathcal{T}^*(\omega_2^L), \sim \mathcal{T}^{**}(\omega_2^L))$. Let $T_n, n \in \omega$ be the trees defined in the proof of (a), when $\kappa = \omega_2^L$. Then n belongs to $0^\#$ iff T_n has a cardinal-preserving, $\mathcal{P}(\omega_1^L)$ -preserving, cofinal branch. Moreover, if P_n is the L -definable forcing for producing such a branch, then for n in $0^\#$ this branch can be required to satisfy the stronger version of (*) in which we require $s \upharpoonright \eta$ to code over L_β a generic for $(P_n)^{L_\beta}$.

We aim to define a new sequence of trees $T_n^*, n \in \omega$. To do so we first define $f_n(\alpha)$ by induction on $\alpha < \omega_2^L$:

- (1) If $f_n(\bar{\alpha})$ has length $\geq \alpha$ for some $\bar{\alpha} < \alpha$ or $L\text{-cof}(\alpha) = \omega$ then $f_n(\alpha)$ is the least element of T_n of length $\geq \alpha$ not in $f_n[\alpha]$.
- (2) If (1) fails then let β be least such that $L_\beta \models ZF^- + \alpha = \omega_2$ and for some least condition $p \in (P_n)^{L_\beta}$ and least name σ in $L_\beta : p \Vdash \sigma$ is a CUB subset

of α disjoint from $f_n[b^G]$, b^G denoting the generic branch through $T_n \upharpoonright \alpha$. If β does not exist then define $f_n(\alpha)$ as in (1). Otherwise choose $f_n(\alpha)$ to be the least element of T_n of length α coding a $(P_n)^{L_\beta}$ -generic extending p .

Now define T_n^* by $\alpha <_{T_n^*} \beta$ iff $f_n(\alpha) <_{T_n} f_n(\beta)$. Suppose that b is a $\mathcal{P}(\omega_1^L)$ -preserving branch through T_n . We claim that $f_n[b] \cap (\text{cof } \omega_1)^L$ is a stationary branch through T_n^* . If not then (as b generically codes a P_n -generic) there is $\beta > \omega_2^L$ satisfying $L_\beta \models \text{ZF}^-$, $p \in (P_n)^{L_\beta}$ and $\sigma \in L_\beta$ such that $p \Vdash \sigma$ is a CUB subset of ω_2^L disjoint from $f_n[b^G]$, where b^G denotes the generic branch through T_n . Choose β, p, σ to be least. By reflection there is $\alpha \in (\text{cof } \omega_1)^L \cap \omega_2^L$ so that $f_n(\alpha)$ is defined as in the last part of (2) above. Choose c to be a branch through T_n generically coding a $(P_n)^{L_\beta}$ -generic extending p , where c extends $f_n(\alpha)$. (This is possible, using the countability of β in $L[0^\#]$, as we can choose α so that $\alpha = \omega_2^L \cap \text{Skolem hull}(\alpha)$ in L_β , thereby enabling the extension from p_{ω_1} to $f_n(\alpha)$ to obey the restraint imposed by $p_{\omega_1}^*$). But now we have a contradiction, as α is a limit of σ^c , a CUB subset of ω_2^L , and $f_n(\alpha)$ lies on c .

If n belongs to $0^\#$ then T_n has a generic branch and hence a cardinal-preserving, $\mathcal{P}(\omega_1^L)$ -preserving branch; by the above, T_n^* has a stationary such branch. If n does not belong to $0^\#$ then T_n has no cardinal-preserving branch and hence T_n^* has no cardinal-preserving branch, as such a branch b would yield the cardinal-preserving branch $f_n[b]$ through T_n . Thus it follows that $0^\# \longrightarrow_L (\mathcal{T}^*(\omega_2^L), \sim \mathcal{T}^{**}(\omega_2^L))$.

The next lemma is from [9], reformulated in terms of our notion of reduction.

Lemma 5 $(\mathcal{T}^*(\omega_2^L), \sim \mathcal{T}^{**}(\omega_2^L)) \longrightarrow_L \mathcal{C}(\omega_2^L)$, where for L -regular κ , $\mathcal{C}(\kappa) = \{X \subseteq \kappa \mid X \in L \text{ and } X \text{ contains a cardinal-preserving CUB subset}\}$.

Proof of Lemma 5 Let T be a tree on ω_2^L of height ω_2^L which is Δ_1 -definable over $L_{\omega_2^L}$ from the parameter ω_1^L . Then there is a gap 1 morass at ω_1^L in L such that:

- (1) To each $\sigma \in \cup\{S_\alpha \mid \alpha \leq \omega_1^L\}$ is associated a tree T_σ on σ .
- (2) For $\sigma \in S_{\omega_1^L}$, $T_\sigma = T \upharpoonright \sigma$.
- (3) For $\sigma <_1 \tau$, $T_\sigma = \pi_{\sigma\tau}^{-1}[T_\tau]$ and for $\sigma <_0 \tau$, $T_\sigma = T_\tau \upharpoonright \sigma$.

(Our notation is as follows: S_α is the α -th level of the morass; $\sigma <_1 \tau$ means that σ lies below τ in the morass tree, with associated morass map $\pi_{\sigma\tau}$; $\sigma <_0 \tau$ means that σ and τ lie on the same morass level and σ is a smaller ordinal than τ .)

To obtain this morass, start with the morass constructed in [7], thinning out $S_{\omega_1^L}$ to consist of σ such that $T \upharpoonright \sigma$ is uniformly $\Delta_1(L_\sigma)$ in parameter ω_1^L . Then define T_σ for $\alpha(\sigma) < \omega_1^L$ to be the tree on σ defined over L_σ with parameter $\omega_1^{L_\sigma}$ in the same way as T is defined over $L_{\omega_2^L}$ with parameter ω_1^L .

For $\alpha < \omega_1^L$ let $\sigma(\alpha)$ be the maximum of S_α . By induction on $\alpha < \omega_1^L$ we define $X_\alpha \subseteq \sigma(\alpha)$. For any $\sigma \in S_\alpha$ and $i < \alpha$, $\sigma(i)$ denotes the unique $\bar{\sigma} <_1 \sigma$ such that $\bar{\sigma} \in S_i$, if it exists. Now let $\langle \beta_i \mid i < \alpha \rangle$ be the least α -sequence in $L_{\sigma(\alpha)}$ of pairwise $T_{\sigma(\alpha)}$ -compatible elements of $\sigma(\alpha)$ such that if $\sigma_i = \beta_i$ -th element of S_α then $\{i < \alpha \mid \text{For all } j < i, \sigma_j(i) \in X_{\alpha(\sigma_j(i))}\} = \emptyset$. Then X_α consists of all $\sigma \in S_\alpha$ such that there is $\bar{\sigma} <_1 \sigma$, $\pi_{\bar{\sigma}\sigma}$ cofinal, $\bar{\sigma} \in X_{\alpha(\bar{\sigma})}$ together with each σ_i . (Ignore the σ_i 's if $\langle \beta_i \mid i < \alpha \rangle$ does not exist.)

Let $X = (\text{cof } \omega_1)^L \cup \{\sigma \in S_{\omega_1^L} \mid \text{There exists } \bar{\sigma} <_1 \sigma, \pi_{\bar{\sigma}\sigma} \text{ cofinal, } \bar{\sigma} \in X_{\alpha(\bar{\sigma})}\}$. If X contains a cardinal-preserving CUB subset C then note that $\sigma, \tau \in \{\beta \mid \text{ordertype}(C \cap \beta) = \beta\} \cap (\text{cof } \omega_1)^L \rightarrow \sigma, \tau$ are T -compatible; otherwise $\{i < \omega_1^L \mid \cup \text{Range } \pi_{\sigma(i)\sigma} \in X \text{ and } \cup \text{Range } \pi_{\tau(i)\tau} \in X\}$ is empty, by construction, in contradiction to the fact that $C \cap \sigma, C \cap \tau$ are CUB in σ, τ of L -cofinality ω_1^L . Therefore T has a cardinal-preserving branch. Conversely, suppose that T has a cardinal-preserving, stationary, $\mathcal{P}(\omega_1^L)$ -preserving branch b . For $\sigma \in S_{\omega_1^L}$, σ of L -cofinality ω_1^L and $i < \omega_1^L$, let σ^i denote $\cup \text{Range } \pi_{\sigma(i)\sigma}$ (when $\sigma(i)$ is defined). By construction of X , if we let $X(\sigma)$ denote $\{i \mid \sigma^i \in X\}$ then $\{X(\sigma) \mid \sigma \in b^*\}$ generates a normal filter \mathcal{F} , where $b^* = \{\sigma \in b \mid \sigma = \text{ordertype}(S_{\omega_1^L} \cap \sigma)\} \cap (\text{cof } \omega_1)^L$. But now there is a cardinal-preserving forcing (over $L[b]$) to add a CUB subset of X , using conditions which are closed, bounded subsets of X , making use of the fact that the $X(\sigma)$ for $\sigma \in b^*$ generate a normal filter to prove ω_2^L -distributivity.

Thus if T belongs to $\mathcal{T}^*(\omega_2^L)$ then X has a cardinal-preserving CUB subset, and if T belongs to $\sim \mathcal{T}^{**}(\omega_2^L)$ then X does not have a cardinal-preserving CUB subset. This proves the Lemma.

Lemma 5 immediately implies: $0^\# \rightarrow_L \mathcal{C}(\omega_2^L)$. Now suppose that κ is an arbitrary L -regular cardinal greater than ω_2^L . We claim: $\mathcal{C}(\omega_2^L) \rightarrow_L \mathcal{C}(\kappa)$. Indeed, let $\langle C_\alpha \mid \alpha \text{ singular in } L \rangle$ be an L -definable \square -sequence; i.e., C_α is cofinal in α of ordertype less than α and if $\bar{\alpha}$ is a limit point of C_α then

$C_{\bar{\alpha}} = C_{\alpha} \cap \bar{\alpha}$ (see [7]). Given a constructible subset X of ω_2^L define $X^* \subseteq \kappa$ as follows: For $\alpha < \kappa$ of L -cofinality at most ω_1^L define $\alpha_0 = \alpha$, $\alpha_1 =$ ordertype of C_{α_0} , $\alpha_2 =$ ordertype of C_{α_1} , \dots until you first reach $\alpha_k = \alpha^*$ less than ω_2^L . Then α belongs to X^* iff this α^* belongs to X . In addition X^* contains all ordinals less than κ of L -cofinality greater than ω_1^L . If X^* contains the cardinal-preserving CUB subset C^* , then X contains a cardinal-preserving CUB subset C , obtained by choosing β to be the ω_2^L -th element of C^* greater than ω_2^L and letting C consist of those α^* obtained from α in a final segment of $C^* \cap C_{\beta}$. And conversely, if X contains a cardinal-preserving CUB subset C then so does X^* , obtained by forcing over $L[C]$ with bounded, closed subsets of X^* . It follows that X contains a cardinal-preserving CUB subset iff X^* does, and therefore $0^{\#} \rightarrow_L \mathcal{C}(\omega_2^L) \rightarrow_L \mathcal{C}(\kappa)$. This completes the proof of (b).

(c) First note that in the proof of (b) we in fact showed

$$0^{\#} \rightarrow_L (\mathcal{C}^*(\omega_2^L), \sim \mathcal{C}^{**}(\omega_2^L)),$$

where $\mathcal{C}^*(\omega_2^L) = \{X \subseteq \omega_2^L \mid X \text{ is } \Delta_1\text{-definable over } L_{\omega_2^L} \text{ from parameter } \omega_1^L \text{ and contains a cardinal-preserving, } \mathcal{P}(\omega_1^L)\text{-preserving CUB subset}\}$, $\mathcal{C}^{**}(\omega_2^L) = \{X \subseteq \omega_2^L \mid X \text{ is } \Delta_1\text{-definable over } L_{\omega_2^L} \text{ from parameter } \omega_1^L \text{ and contains a cardinal-preserving CUB subset}\}$. Now define

$$\mathcal{A}(\omega_1^L) = \{Y \mid Y \in L, Y \subseteq \omega_1^L \text{ and } Y = \Lambda(R) \cap \omega_1^L \text{ for some cardinal-preserving real } R\},$$

where $\Lambda(R) =$ the class of R -admissible ordinals.

Lemma 6 $(\mathcal{C}^*(\omega_2^L), \sim \mathcal{C}^{**}(\omega_2^L)) \rightarrow_L \mathcal{A}(\omega_1^L)$.

Proof of Lemma 6 Let X be a subset of the interval (ω_1^L, ω_2^L) which is Δ_1 -definable over $L_{\omega_2^L}$ from the parameter ω_1^L . As in the proof of Lemma 5, we may choose a constructible morass such that to each $\sigma \in \cup\{S_{\alpha} \mid \alpha \leq \omega_1^L\}$ is assigned $X_{\sigma} \subseteq \sigma$ such that:

- (1) For $\sigma \in S_{\omega_1^L}$, $X_{\sigma} = X \cap \sigma$.
- (2) $\sigma <_0 \tau \rightarrow X_{\sigma} = X_{\tau} \cap \sigma$, $\sigma <_1 \tau \rightarrow X_{\sigma} = \pi_{\sigma\tau}^{-1}[X_{\tau}]$.

We set $Y = \{\eta \mid \text{For some countable morass point } \sigma, \eta \in [\alpha(\sigma), \sigma^*] \text{ is } X_{\sigma^*}\text{-admissible and belongs to } X_{\sigma^*} \cup \{\alpha(\sigma), \sigma^*\}, \text{ where } \sigma^* = \max S_{\alpha(\sigma)}\}$. Now consider the forcing P whose conditions are of the form (s, s^*, t) where:

- (i) $s : |s| \rightarrow 2, |s| < \omega_1^L$ and $\eta \leq |s| \rightarrow (\eta \text{ is } s \upharpoonright \eta\text{-admissible iff } \eta \text{ belongs to } Y)$.
- (ii) $t : |t| \rightarrow 2, |t| < \omega_2^L$ and $\omega_1^L \leq \eta \leq |t| \rightarrow (\eta \text{ is } t \upharpoonright \eta\text{-admissible iff } \eta \text{ is } X\text{-admissible and belongs to } X \cup \{\omega_1^L\})$.
- (iii) s^* is restraint for coding $G_{\omega_1^L}$ by G_ω .

P factors as $P_{\omega_1^L} * Q$ where $P_{\omega_1^L}$ consists of t as in (ii) above and $Q = P_{\omega_1^L}^G$ codes the generic $G = G_{\omega_1^L}$ for $P_{\omega_1^L}$ using conditions s as in (i), with restraint. And $P_{\omega_1^L} \Vdash Q$ is ω_1^L -distributive and ω_2^L -cc. Thus P is ω_2^L -preserving if $P_{\omega_1^L}$ is ω_2^L -distributive. If X contains a $\mathcal{P}(\omega_1^L)$ -preserving CUB subset then the ω_2^L -distributivity of $P_{\omega_1^L}$ holds in such a $\mathcal{P}(\omega_1^L)$ -preserving extension, and hence in L . And the P -generic $G_\omega \subseteq \omega_1^L$ can be coded by a cardinal-preserving real preserving admissibles, proving that Y belongs to $\mathcal{A}(\omega_1^L)$. Conversely, if $\Lambda(R) \cap \omega_1^L = Y$ for some cardinal-preserving real R then X contains the cardinal-preserving CUB subset of ω_2^L consisting of all $\alpha < \omega_2^L$ such that $L_\alpha[R]$ is Σ_1 -elementary in $L_{\omega_2^L}[R]$. This proves the Lemma, and therefore the reduction $0^\# \rightarrow_L \mathcal{A}(\omega_1^L)$.

Now notice that in the above we produced a constructible sequence $\langle Y_n \mid n \in \omega \rangle$ such that n belongs to $0^\#$ iff $Y_n = \Lambda(R) \cap \omega_1^L$ for some cardinal-preserving real R , and in addition the sets Y_n are Δ_1 over $L_{\omega_2^L}$ from the parameter ω_1^L . Now for any uncountable L -cardinal κ define Y_n^κ over L_κ in the same way that Y_n is defined over $L_{\omega_2^L}$. Then n belongs to $0^\#$ iff $Y_n^\kappa = \Lambda(R) \cap \kappa$ for some cardinal-preserving real R . The reduction $0^\# \rightarrow \mathcal{A}(\kappa)$ therefore holds for every uncountable L -cardinal κ . This proves (c).

(d) Let $\mathcal{D}(\kappa)$ denote the set described in the statement of (d). We show: $(\mathcal{C}^*(\omega_2^L), \sim \mathcal{C}^{**}(\omega_2^L)) \rightarrow_L \mathcal{D}(\omega_3^L)$. Suppose that X is a subset of the interval (ω_1^L, ω_2^L) which is Δ_1 -definable over $L_{\omega_2^L}$ from parameter ω_1^L . If X contains a cardinal-preserving, $\mathcal{P}(\omega_1^L)$ -preserving CUB subset C then in $L[C]$ the set $X \cup \{\omega_2^L\}$ obeys the Condensation Condition at ω_1^L (see [5], Section 8.1). And in L we may define $Y \subseteq \omega_1^L$ such that $Y \cup \{\omega_1^L\} \cup X \cup \{\omega_2^L\}$ obeys

the Condensation Condition at ω in L , and hence in $L[C]$. Thus by the Δ_1 -Coding Theorem (Section 8.1 of [5]), $Y \cup \{\omega_1^L\} \cup X \cup \{\omega_2^L\}$ belongs to $\mathcal{D}(\omega_3^L)$. Conversely, if $Y \cup \{\omega_1^L\} \cup X \cup \{\omega_2^L\}$ belongs to $\mathcal{D}(\omega_3^L)$ for some $Y \subseteq \omega_1^L$ then X contains a cardinal-preserving CUB subset. Thus we have the desired reduction. As in the last part of the proof of (c), we also obtain the reduction $0^\# \rightarrow_L \mathcal{D}(\kappa)$ for arbitrary L -cardinals $\kappa > \omega_2^L$. This completes the proof of Theorem 3. \square

Proof of Theorem 4 This follows from the Condensation Condition in Section 8.1 of [5]. Suppose that A is a constructible subset of ω_2^L . We say that an extensional set x of ordinals *preserves* A if $\langle \bar{x}, \in, A \cap \bar{x} \rangle$ is isomorphic to $\langle x, \in, A \cap x \rangle$, where \bar{x} denotes the transitive collapse of x . Suppose that A satisfies the condition

(*) In L , the set of countable subsets of ω_2 which preserve A is stationary.

Then A obeys the Condensation Condition at ω and therefore by the Δ_1 -Coding Theorem (Theorem 8.3 of [5]), A is the intersection with ω_2^L of a class which is Δ_1 -definable over $L[R]$ without parameters, for some cardinal-preserving real R . Thus A belongs to $\mathcal{D}(\omega_2^L)$. Conversely, (*) implies the Condensation Condition at ω and therefore the latter provides a constructible criterion for a set A to belong to $\mathcal{D}(\omega_2^L)$. (This argument is special to ω_2^L , as at cardinals greater than ω_2^L , (*) is strictly weaker than the Condensation Condition.) \square

Some Open Questions

1. Is there a reduction $\mathcal{T}(\omega_2^L) \rightarrow_L \mathcal{C}(\omega_2^L)$? If so, the arguments of this paper could perhaps be simplified.

2. What happens if we weaken cardinal-preservation to the preservation of only some cardinals? For example, is the set

$$\mathcal{C}'(\omega_3^L) = \{X \in L \mid X \subseteq \omega_3^L \text{ and } X \text{ has a CUB subset in an extension of } L \text{ which preserves } \omega_1^L, \omega_3^L\}$$

constructible?

3. What is the situation with the tree problem at κ , when trees on κ are required to have levels of size less than κ ?

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