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REPORT No. 2, 2000/2001

ISSN 1103-467X
ISRN IML-R- -2-00/01- -SE

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ABSTRACT. We prove that the classification problem for torsion-free abelian groups of rank three is not Borel reducible to that for torsion-free abelian groups of rank two.

1. INTRODUCTION

This paper is a contribution to the project [9, 8, 1, 13] of explaining why no satisfactory system of complete invariants has yet been found for the torsion-free abelian groups of finite rank $n \geq 2$. Recall that, up to isomorphism, the torsion-free abelian groups of rank n are exactly the additive subgroups of the n-dimensional vector space $\mathbb{Q}^n$ which contain n linearly independent elements. Thus the collection of torsion-free abelian groups of rank $1 \leq r \leq n$ can be naturally identified with the set $S(\mathbb{Q}^n)$ of all nontrivial additive subgroups of $\mathbb{Q}^n$. In 1937, Baer [3] solved the classification problem for the class $S(\mathbb{Q})$ of rank one groups as follows.

Let $\mathbb{P}$ be the set of primes. If G is a torsion-free abelian group and $0 \neq x \in G$, then the *p-height* of x is defined to be

$$h_x(p) = \sup\{n \in \mathbb{N} \mid \text{ There exists } y \in G \text{ such that } p^n y = x\ \} \in \mathbb{N} \cup \{\infty\};$$

and the *characteristic* $\chi(x)$ of x is defined to be the function

$$\langle h_x(p) \mid p \in \mathbb{P} \rangle \in (\mathbb{N} \cup \{\infty\})^{\mathbb{P}}.$$

Two functions $\chi_1, \chi_2 \in (\mathbb{N} \cup \{\infty\})^{\mathbb{P}}$ are said to be *similar* or to *belong to the same type*, written $\chi_1 \equiv \chi_2$, iff

(a) $\chi_1(p) = \chi_2(p)$ for almost all primes p; and

(b) if $\chi_1(p) \neq \chi_2(p)$, then both $\chi_1(p)$ and $\chi_2(p)$ are finite.

Research partially supported by NSF Grants.

Clearly $\equiv$ is an equivalence relation on $(\mathbb{N} \cup \{\infty\})^{\mathbb{P}}$. If G is a torsion-free abelian group and $0 \neq x \in G$, then the *type* $\tau(x)$ of x is defined to be the $\equiv$-equivalence class containing the characteristic $\chi(x)$.

Now suppose that $G \in S(\mathbb{Q})$ is a rank one group. Then it is easily checked that $\tau(x) = \tau(y)$ for all $0 \neq x, y \in G$. Hence we can define the *type* $\tau(G)$ of G to be $\tau(x)$, where x is any non-zero element of G. In [3], Baer proved that $\tau(G)$ is a complete invariant for the isomorphism problem for $S(\mathbb{Q})$.

Theorem 1.1 (Baer [3]). *If G, $H \in S(\mathbb{Q})$, then $G \cong H$ iff $\tau(G) = \tau(H)$.*

However, the situation is much less satisfactory in the case of the torsion-free abelian groups of rank $n \geq 2$. In the late 1930s, Kurosh [15] and Malcev [16] found complete invariants for these groups consisting of equivalence classes of infinite sequences $\langle M_p \mid p \in \mathbb{P} \rangle$ of matrices, where each $M_p \in GL_n(\mathbb{Q}_p)$. However, as Fuchs [7, Section 93] remarks, the associated equivalence relation is so complicated that the problem of deciding whether two sequences are equivalent is as difficult as that of deciding whether the corresponding groups are isomorphic. It is natural to ask whether the classification problem for $S(\mathbb{Q}^n)$ is genuinely more difficult when $n \geq 2$; or whether, on the contrary, there exists an "explicit" map $f : S(\mathbb{Q}^n) \to S(\mathbb{Q})$ which reduces the classification problem for $S(\mathbb{Q}^n)$ to that for $S(\mathbb{Q})$; i.e. which has the property that $A \cong B$ iff $f(A) \cong f(B)$. To give a precise formulation of this question, we need to make use of the notion of Borel reducibility.

Let X be a standard Borel space; i.e. a Polish space equipped with its Borel structure. Then a *Borel equivalence relation* on X is an equivalence relation $E \subseteq X^2$ which is a Borel subset of X^2. If E, F are Borel equivalence relations on the standard Borel spaces X, Y respectively, then we say that E is *Borel reducible* to F and write $E \leq_B F$ if there exists a Borel function $f : X \to Y$ such that xEy iff $f(x)Ff(y)$. We say that E and F are *Borel bireducible* and write $E \sim_B F$ if both $E \leq_B F$ and $F \leq_B E$. Finally we write $E <_B F$ if both $E \leq_B F$ and $F \nleq_B E$. Most of the Borel equivalence relations that we shall consider in this paper arise from group actions as follows. Let G be a locally compact second countable group. Then a *standard Borel G-space* is a standard Borel space X equipped with a Borel action $(g, x) \mapsto g.x$ of G on X. The corresponding G-orbit equivalence relation on X, which we shall denote by E_G^X, is a Borel equivalence relation. In fact, by

Kechris [11], E_G^X is Borel bireducible with a *countable Borel equivalence relation*; i.e. a Borel equivalence relation E such that every E-equivalence class is countable. Conversely, by Feldman-Moore [6], if E is an arbitrary countable Borel equivalence relation on the standard Borel space X, then there exists a countable group G and a Borel action of G on X such that $E = E_G^X$.

To see how the classification problem for torsion-free abelian groups fits into this context, note that $S(\mathbb{Q}^n)$ is a Borel subset of the Polish space $\mathcal{P}(\mathbb{Q}^n)$ of all subsets of $\mathbb{Q}^n$ and so $S(\mathbb{Q}^n)$ is a standard Borel space. (Here we are identifying $\mathcal{P}(\mathbb{Q}^n)$ with the space $2^{\mathbb{Q}^n}$ of all functions $h : \mathbb{Q}^n \to \{0,1\}$ equipped with the product topology.) Furthermore, $S(\mathbb{Q}^n)$ is a standard Borel $GL_n(\mathbb{Q})$-space under the action induced from the natural action of $GL_n(\mathbb{Q})$ on the vector space $\mathbb{Q}^n$; and it is easily checked that if $A, B \in S(\mathbb{Q}^n)$, then $A \cong B$ iff there exists an element $\varphi \in GL_n(\mathbb{Q})$ such that $\varphi[A] = B$. Thus the isomorphism relation on $S(\mathbb{Q}^n)$ is a countable Borel equivalence relation.

Notation 1.2. $\cong_n$ denotes the isomorphism relation on $S(\mathbb{Q}^n)$.

It is clear that $(\cong_n) \leq_B (\cong_{n+1})$ for all $n \geq 1$; and our earlier question on the complexity of the classification problem for $S(\mathbb{Q}^n)$ can be rephrased as the question of whether $(\cong_1) <_B (\cong_n)$ when $n \geq 2$. In order to explain the solution of this problem and to be able to formulate the main open problems in this area, it is first necessary to give a brief account of some of the general theory of countable Borel equivalence relations. (A detailed development of the theory can be found in Jackson-Kechris-Louveau [10].)

The least complex countable Borel equivalence relations are those which are *smooth*; i.e. those countable Borel equivalence relations E on a standard Borel space X for which there exists a Borel function $f : X \to Y$ into a standard Borel space Y such that xEy iff $f(x) = f(y)$. Next in complexity come those countable Borel equivalence relations E such that E is Borel bireducible with the *Vitali equivalence relation* E_0 defined on $2^{\mathbb{N}}$ by xE_0y iff $x(n) = y(n)$ for almost all n. By Dougherty-Jackson-Kechris [5], if E is a countable Borel equivalence relation on a standard Borel space X, then the following three properties are equivalent:

(1) $E \leq_B E_0$.

(2) E is *hyperfinite*; i.e. there exists an increasing sequence

$$F_0 \subseteq F_1 \subseteq \cdots \subseteq F_n \subseteq \cdots$$

of finite Borel equivalence relations on X such that $E = \bigcup_{n \in \mathbb{N}} F_n$. (Here an equivalence relation F is said to be *finite* iff every F-equivalence class is finite.)

(3) There exists a Borel action of $\mathbb{Z}$ on X such that $E = E_{\mathbb{Z}}^X$.

It is easily checked that the similarity relation $\equiv$ on the space $(\mathbb{N} \cup \{\infty\})^{\mathbb{P}}$ of characteristics is Borel bireducible with E_0. Thus we obtain the following characterisation of the complexity of the isomorphism problem for $S(\mathbb{Q})$.

Theorem 1.3 (Folklore). $(\cong_1) \sim_B E_0$.

It turns out that there is also a most complex countable Borel equivalence relation E_∞, which is *universal* in the sense that $F \leq_B E_\infty$ for every countable Borel equivalence relation F, and that $E_0 <_B E_\infty$. (Clearly this universality property uniquely determines E_∞ up to Borel bireducibility.) E_∞ has a number of natural realisations in many areas of mathematics, including algebra, topology and recursion theory. (See Jackson-Kechris-Louveau [10].) For example, E_∞ is Borel bireducible to both the isomorphism relation for finitely generated groups [21] and the isomorphism relation for fields of finite transcendence degree [22].

In [9], Hjorth-Kechris conjectured that $(\cong_n) \sim_B E_\infty$ for all $n \geq 2$. Of course, if true, this would explain the failure to find a satisfactory system of complete invariants for the torsion-free abelian groups of rank $n \geq 2$, since nobody expects such a system to exist for the class of finitely generated groups. In [8], Hjorth provided some evidence for this conjecture by proving that $E_0 <_B (\cong_n)$ for all $n \geq 2$. (For $n \geq 3$, Hjorth proved the stronger result that $\cong_n$ is not treeable. More recently, Kechris [13] has shown that $\cong_2$ is also not treeable. See [9] or [10] for a discussion of the notion of treeability.) Later in [1], Adams-Kechris used Zimmer's superrigidity theorem for cocycles [24, Theorem 5.2.5] to prove the intriguing result that

$$(\cong_1^*) <_B (\cong_2^*) <_B \cdots <_B (\cong_n^*) <_B \cdots$$

where $(\cong_n^*)$ is the restriction of the isomorphism relation to the class of *rigid* torsion-free abelian groups $A \in S(\mathbb{Q}^n)$. Here an abelian group A is said to be rigid if its only automorphism are the obvious ones: $a \mapsto a$ and $a \mapsto -a$. In particular, none of the relations $\cong_n^*$ is a universal countable Borel equivalence relation. It was not clear whether or not the Adams-Kechris result provided further evidence for the Hjorth-Kechris conjecture, since very little was known concerning the relationship between $\cong_n^*$ and $\cong_m$ for $n, m \geq 1$. Of course, it is clear that $(\cong_n^*) \leq_B (\cong_n)$ for all $n \geq 1$; and using Theorem 1.1, it is easily seen that $(\cong_1^*) \sim_B E_0$ and so $(\cong_1^*) \sim_B (\cong_1)$. But, apart from these easy observations, essentially nothing else was known. The main result of this paper says that $(\cong_3^*) \not\leq_B (\cong_2)$. Thus $\cong_2$ is not a universal countable Borel equivalence relation and so the Hjorth-Kechris conjecture is false.

Theorem 1.4. $(\cong_3^*) \not\leq_B (\cong_2)$.

As an immediate consequence, we obtain that the classification problem for $S(\mathbb{Q}^3)$ is strictly more complex than that for $S(\mathbb{Q}^2)$.

Corollary 1.5. $(\cong_2) <_B (\cong_3)$.

Theorem 1.4 is an easy consequence of Theorem 1.6. But before we can state Theorem 1.6, we need to recall some notions from ergodic theory and group theory. Let G be a locally compact second countable group and let X be a standard Borel G-space. Throughout this paper, a probability measure on X will always mean a Borel probability measure; i.e. a measure which is defined on the collection of Borel subsets of X. The probability measure μ on X is said to be *nonatomic* if $\mu(\{x\}) = 0$ for all $x \in X$; and μ is said to be *G-invariant* iff $\mu(g[A]) = \mu(A)$ for every $g \in G$ and Borel subset $A \subseteq X$. The G-invariant probability measure μ is *ergodic* iff for every G-invariant Borel subset $A \subseteq X$, either $\mu(A) = 0$ or $\mu(A) = 1$. It is well-known that the following two properties are equivalent:

(i) μ is ergodic.

(ii) If Y is a standard Borel space and $f : X \to Y$ is a G-invariant Borel function, then there exists a G-invariant Borel subset $M \subseteq X$ with $\mu(M) = 1$ such that $f \upharpoonright M$ is a constant function.

Let G be a locally compact second countable group and let $\pi : G \to U(\mathcal{H})$ be a unitary representation of G on the separable Hilbert space $\mathcal{H}$. Then π *almost* *admits invariant vectors* if for every $\varepsilon > 0$ and every compact subset $K \subseteq G$, there exists a unit vector $v \in \mathcal{H}$ such that $||\pi(g).v - v|| < \varepsilon$ for all $g \in K$. We say that G is a *Kazhdan group* if for every unitary representation π of G, if π almost admits invariant vectors, then π has a non-zero invariant vector. If G is a connected semisimple $\mathbb{R}$-group, each of whose almost $\mathbb{R}$-simple factors has $\mathbb{R}$-rank at least 2, and Γ is a lattice in G, then Γ is a Kazhdan group. (For example, see Margulis [17] or Zimmer [24].) In particular, $SL_3(\mathbb{Z})$ is a Kazhdan group.

For later use, recall that a locally compact second countable group G is *amenable* if there exists a finitely additive G-invariant probability measure $\nu : \mathcal{P}(G) \to [0,1]$ defined on every subset of G. During the proof of Theorem 1.6, we shall make use of the fact that if G is either soluble or abelian-by-finite, then G is amenable. (For example, see Wagon [23, Theorem 10.4].)

In the first three sections of this paper, we shall only discuss countable groups equipped with the discrete topology. In Section 4, we shall also need to consider various linear algebraic groups $G(K) \leqslant GL_n(K)$, where K is either $\mathbb{R}$ or a finite extension of the field $\mathbb{Q}_p$ of p-adic numbers for some prime p. In this case, $G(K)$ is a locally compact second countable group with respect to the *Hausdorff topology*; i.e. the topology obtained by restricting the natural topology on K^{n^2} to $G(K)$. Any topological notions concerning the group $G(K)$ will always refer to the Hausdorff topology.

Theorem 1.6. *Let Γ be a countable Kazhdan group and let X be a standard Borel Γ-space with an invariant ergodic probability measure μ. If $f : X \to S(\mathbb{Q}^2)$ is a Borel function such that $x E_\Gamma^X y$ implies $f(x) \cong_2 f(y)$, then there exists a Γ-invariant Borel subset M with $\mu(M) = 1$ such that f maps M into a single $\cong_2$-class.*

Proof of Theorem 1.4. Let $S(\mathbb{Q}^3, \mathbb{Z}^3)$ be the Borel set consisting of those $G \in S(\mathbb{Q}^3)$ such that $\mathbb{Z}^3 \leqslant G$. Then $S(\mathbb{Q}^3, \mathbb{Z}^3)$ is invariant under the action of the subgroup $SL_3(\mathbb{Z})$ of $GL_3(\mathbb{Q})$; and building upon earlier work of Hjorth [8], Adams-Kechris [1, Section 6] have shown that there exists an $SL_3(\mathbb{Z})$-invariant Borel subset X of $S(\mathbb{Q}^3, \mathbb{Z}^3)$ with the following properties.

(i) Each $G \in X$ is rigid.

(ii) There exists an $SL_3(\mathbb{Z})$-invariant ergodic nonatomic probability measure μ on X.

Suppose that $(\cong_3^*) \leq_B (\cong_2)$. Then there exists a Borel function $f : X \to S(\mathbb{Q}^2)$ such that $G \cong_3^* H$ iff $f(G) \cong_2 f(H)$. If G is $E^X_{SL_3(\mathbb{Z})}$-equivalent to H, then $G \cong_3^* H$ and so $f(G) \cong_2 f(H)$. Since $SL_3(\mathbb{Z})$ is a Kazhdan group, Theorem 1.6 implies that there exists an $SL_3(\mathbb{Z})$-invariant Borel subset M with $\mu(M) = 1$ such that f maps M into a single $\cong_2$-class C. But clearly $f^{-1}[C]$ consists of only countably many $E^X_{SL_3(\mathbb{Z})}$-classes, which contradicts the fact that μ is nonatomic. Hence $(\cong_3^*) \not\leq_B (\cong_2)$. □

This paper is organised as follows. In Section 2, we shall discuss the notion of a cocycle of a group action and state the two cocycle reduction results which are needed in the proof of Theorem 1.6. In Section 3, we shall proof Theorem 1.6; and in Section 4, we shall prove our main cocycle reduction result.

Finally we shall address the question of whether it is likely that $(\cong_n) <_B (\cong_{n+1})$ for all $n \geq 1$. Hjorth's result that $(\cong_1) <_B (\cong_2)$ depends essentially upon the fact that $SL_2(\mathbb{Z})$ is nonamenable, while $GL_1(\mathbb{Q}) = \mathbb{Q}^*$ is amenable. In this paper, the proof that $(\cong_2) <_B (\cong_3)$ is based upon the fact that $SL_3(\mathbb{Z})$ is a Kazhdan group, while $GL_2(\mathbb{Q})$ does not contain any infinite Kazhdan subgroups. However, we could also have based our proof upon Zimmer's superrigidity theorem for cocycles [24, Theorem 5.2.5], which can be used to distinguish between $SL_n(\mathbb{Z})$-spaces and $SL_{n+1}(\mathbb{Z})$-spaces for all $n \geq 2$. In fact, the main obstruction to proving that $(\cong_n) <_B (\cong_{n+1})$ for $n \geq 3$ lies in the field of abelian group theory. The proof of Theorem 1.6 makes heavy use of Król's analysis [14] of the automorphism groups and endomorphism rings of the torsion-free abelian groups of rank two; and no such analysis has yet been made for the groups of rank $n \geq 3$.

Conjecture 1.7. $(\cong_n) <_B (\cong_{n+1})$ for all $n \geq 1$.

2. Cocycles

In this section, we shall discuss the notion of a cocycle of a group action and state the two cocycle reduction results which are needed in the proof of Theorem 1.6. (Clear accounts of the theory of cocycles can be found in Zimmer [24] and Adams-Kechris [1]. In particular, Adams-Kechris [1, Section 2] provides a convenient introduction to the basic techniques and results in this area, written for the non-expert in the ergodic theory of groups.) Let Γ be a countable group and let X be a standard Borel Γ-space with an invariant probability measure μ.

Definition 2.1. If H is a locally compact second countable group, then a Borel function $\alpha : \Gamma \times X \to H$ is called a *cocycle* if for all $g, h \in \Gamma$ and $x \in X$,

$$\alpha(hg, x) = \alpha(h, g.x)\alpha(g, x).$$

(Since we shall only be considering the case when Γ is a countable group, we can work with strict cocycles throughout this paper. For example, see the discussion on page 67 of Zimmer [24].)

In this paper, cocycles will usually arise in the following fashion. Suppose that H is also a countable group and that Y is a standard Borel H-space. Let $f : X \to Y$ be a Borel function such that $x E_\Gamma^X y$ implies $f(x) E_H^Y f(y)$. If H acts freely on Y, then we can define a Borel cocycle $\alpha : \Gamma \times X \to H$ by letting $\alpha(g, x)$ be the *unique* element of H such that

$$\alpha(g, x).f(x) = f(g.x).$$

Suppose now that $B : X \to H$ is a Borel function and that $f' : X \to Y$ is defined by $f'(x) = B(x).f(x)$. Then $x E_\Gamma^X y$ also implies $f'(x) E_H^Y f'(y)$; and the corresponding cocycle $\alpha' : \Gamma \times X \to H$ satisfies

$$\alpha'(g, x) = B(g.x)\alpha(g, x)B(x)^{-1}$$

for all $g \in \Gamma$ and $x \in X$. This observation motivates the following definition.

Definition 2.2. Let H be a locally compact second countable group. Then the cocycles $\alpha, \beta : \Gamma \times X \to H$ are *equivalent*, written $\alpha \sim \beta$, iff there exists a Borel function $B : X \to H$ such that for all $g \in \Gamma$,

$$\beta(g, x) = B(g.x)\alpha(g, x)B(x)^{-1} \quad \mu\text{-a.e.}$$

A cocycle reduction result says that under suitable hypotheses, every cocycle $\alpha : \Gamma \times X \to H$ is equivalent to a cocycle β such that $\beta[\Gamma \times X]$ is contained in a "small" subgroup of H. To see why this might be useful, we shall continue our discussion of the example which was introduced just before Definition 2.2. Suppose now that $\beta[\Gamma \times X]$ is the identity subgroup of H. Then the associated Borel function $f' : X \to Y$ is Γ-invariant. Hence if μ is an ergodic measure on X, then there exists a Γ-invariant Borel subset M with $\mu(M) = 1$ such that $f' \restriction M$ is a constant function; and this implies that f maps M into a single E_H^Y-class. With a little more effort, we can reach the same conclusion if we only assume that $\beta[\Gamma \times X]$ is contained in a finite subgroup of H.

The proof of Theorem 1.6 will be based on the following cocycle reduction result, which we shall prove in Section 4.

Theorem 2.3. *Let Γ be a countable Kazhdan group and let X be a standard Borel Γ-space with an invariant ergodic probability measure μ. Then for every Borel cocycle $\alpha : \Gamma \times X \to PGL_2(\mathbb{Q})$, there exists an equivalent cocycle γ such that $\gamma[\Gamma \times X]$ is contained in a finite subgroup of $PGL_2(\mathbb{Q})$.*

At this point, we can explain the strategy of the proof of Theorem 1.6. So suppose that Γ is a countable Kazhdan group and that X is a standard Borel Γ-space with an invariant ergodic probability measure μ. Let $f : X \to S(\mathbb{Q}^2)$ be a Borel function such that $x E_\Gamma^X y$ implies $f(x) \cong_2 f(y)$. Because $GL_2(\mathbb{Q})$ does not act freely on $S(\mathbb{Q}^2)$, initially we are unable to define a corresponding cocycle $\alpha : \Gamma \times X \to GL_2(\mathbb{Q})$. Instead we attempt to find a Borel subset X_0 with $\mu(X_0) = 1$ such that $\mathrm{Aut}(f(x))$ is a *fixed* subgroup L of $GL_2(\mathbb{Q})$ for all $x \in X_0$. If we succeed, then $\cong_2 \restriction f[X_0]$ is induced by a free action of the quotient group

$$H = N_{GL_2(\mathbb{Q})}(L)/L;$$

and we can define a corresponding cocycle $\alpha : \Gamma \times X \to H$. Then this case can be dealt with using Theorem 2.3. (At first glance, it might appear that we require a whole series of cocycle reduction results, one for each of the possible groups L. But fortunately we can get by with Theorem 2.3.) On the other hand, in those cases when we fail, it turns out that there exists a Borel subset X_0 with $\mu(X_0) = 1$ such that $\cong_2 \restriction f[X_0]$ is induced by an action of a countable amenable group H. Then the following result deals with this case.

Theorem 2.4. *Let Γ be a countable Kazhdan group and let X be a standard Borel Γ-space with an invariant ergodic probability measure μ. Suppose that H is a countable amenable group and that Y is a standard Borel H-space. If $f : X \to Y$ is a Borel function such that $x E_\Gamma^X y$ implies $f(x) E_H^Y f(y)$, then there exists a Γ-invariant Borel subset M with $\mu(M) = 1$ such that f maps M into a single E_H^Y-class.*

Proof. By Connes-Feldman-Weiss [4], if H is a countable amenable group and Y is a standard Borel H-space, then for every probability measure ν on Y, there exists an H-invariant Borel subset $B \subseteq Y$ with $\nu(B) = 1$ such that $E_H^Y \restriction B$ is hyperfinite. (In [4], the result is only stated for the case when ν is H-quasi-invariant; i.e. when $H.N$ is ν-null for every ν-null Borel subset $N \subseteq Y$. However, as Kechris points out in [12], the result is easily seen to hold for an arbitrary probability measure ν. To see this, let $H = \{h_n \mid n \geq 1\}$ and consider the probability measure ν^* on Y defined by

$$\nu^*(A) = \sum_{n=1}^{\infty} \nu(h_n.A)/2^n.$$

Then ν^* is an H-quasi-invariant probability measure which agrees with ν on every H-invariant Borel set.) In particular, let $\nu = f\mu$ be the probability measure defined on Y by $\nu(A) = \mu(f^{-1}(A))$ for each Borel subset $A \subseteq Y$; and let B be an H-invariant Borel subset B with $\nu(B) = 1$ such that $E_H^Y \restriction B$ is hyperfinite. Let $X_0 = f^{-1}(B)$. Then X_0 is a Γ-invariant subset of X with $\mu(X_0) = 1$. Consider the Borel function $(f \restriction X_0) : X_0 \to B$. By Hjorth-Kechris [9, Theorem 10.5], since Γ is a Kazhdan group and $E_H^Y \restriction B$ is hyperfinite, there exists a Γ-invariant Borel subset $M \subseteq X_0$ with $\mu(M) = 1$ such that f maps M into a single E_H^Y-class. □

3. Torsion-free abelian groups of rank two

In this section, we shall prove Theorem 1.6. For each $\ell = 1, 2$, let $S_\ell(\mathbb{Q}^2)$ be the $GL_2(\mathbb{Q})$-invariant Borel subset consisting of the groups $G \in S(\mathbb{Q}^2)$ of rank ℓ. In the proof of Theorem 1.6, we shall quickly reduce to the case when $f : X \to S_2(\mathbb{Q}^2)$. Following Król [14], our analysis of the groups $G \in S_2(\mathbb{Q}^2)$ will split into various cases depending on the structure of certain invariants $T(G)$ and $\mathcal{C}(G)$, which we shall now describe. So let $G \in S_2(\mathbb{Q}^2)$ be a torsion-free abelian group of rank two. Then

$$T(G) = \{\mathfrak{a} \mid \text{ There exists } 0 \neq a \in G \text{ such that } \tau(a) = \mathfrak{a}\}$$

denotes the set of types which are realised in G. If $0 \neq a \in G$, then the *pure subgroup of G generated by a* consists of those elements $c \in G$ such that there exist k, $\ell \in \mathbb{Z}$ with $k \neq 0$ such that $kc = \ell a$. Now let a, $b \in G$ be a basis of G and let A,

B be the pure subgroups of G generated by a, b respectively. Then $G/(A+B)$ is a torsion group. In fact, by Król 2.23 [14],

$$G/(A+B) \cong \bigoplus_{p\in\mathbb{P}} C(t_p)$$

for some $t_p \in \mathbb{N} \cup \{\infty\}$, where

(a) if $t_p < \infty$, then $C(t_p)$ is a cyclic group of order p^{t_p}; and
(b) if $t_p = \infty$, then $C(t_p)$ is a Prüfer p-group; i.e. an infinite locally cyclic p-group.

The characteristic χ of $G/(A+B)$ is defined to be the function $\chi \in (\mathbb{N}\cup\{\infty\})^{\mathbb{P}}$ such that $\chi(p) = t_p$; and we say that the basis a, b of G realises the triple

$$\langle \mathfrak{a}, \mathfrak{b}, \chi\rangle,$$

where $\mathfrak{a} = \tau(a)$ and $\mathfrak{b} = \tau(b)$. We define $\mathcal{C}(G)$ to be the set of all such triples which are realised by some basis of G.

Lemma 3.1. *Suppose that $G \in S_2(\mathbb{Q}^2)$ and that $\mathfrak{a}$, $\mathfrak{b} \in T(G)$ are two fixed (not necessarily distinct) types. If χ_1, χ_2 are characteristics such that $\langle \mathfrak{a}, \mathfrak{b}, \chi_1\rangle$, $\langle \mathfrak{a}, \mathfrak{b}, \chi_2\rangle \in \mathcal{C}(G)$, then χ_1 and χ_2 belong to the same type.*

Proof. This is an immediate consequence of Theorems 2.25, 2.27 and 2.28 of Król [14]. □

Lemma 3.2. *Let $\mathfrak{a}$, $\mathfrak{b}$ be two fixed (not necessarily distinct) types. Then there only exist countably many groups $G \in S_2(\mathbb{Q}^2)$ such that $\langle \mathfrak{a}, \mathfrak{b}, \chi\rangle \in \mathcal{C}(G)$ for some characteristic χ belonging to the zero-type. (Here the* zero-type *is the type which contain the characteristic $\vartheta \in (\mathbb{N}\cup\{\infty\})^{\mathbb{P}}$ such that $\vartheta(p) = 0$ for all $p \in \mathbb{P}$.)*

Proof. Let A, B be torsion-free abelian groups of rank one such that $\tau(A) = \mathfrak{a}$ and $\tau(B) = \mathfrak{b}$. Then each such group G can be realised up to isomorphism as an extension

$$0 \to A \oplus B \to G \to F \to 0$$

of $A \oplus B$ by a suitably chosen finite abelian group F. Fix a finite abelian group F and let $F = \bigoplus_{i=1}^{k} C_i$ be a decomposition of F into a direct sum of cyclic groups C_i

of order m_i. Then by Theorems 7.14 and 7.17 [19],

$$\begin{aligned}\mathrm{Ext}_{\mathbb{Z}}(F, A \oplus B) &\cong \prod_{i=1}^{k} \mathrm{Ext}_{\mathbb{Z}}(C_i, A \oplus B) \\ &\cong \prod_{i=1}^{k} (A/m_iA) \oplus (B/m_iB).\end{aligned}$$

Thus $\mathrm{Ext}_{\mathbb{Z}}(F, A \oplus B)$ is a finite group; and the result follows easily. □

If $\mathfrak{a}$, $\mathfrak{b}$ are types, then we write $\mathfrak{a} \leq \mathfrak{b}$ iff there exists characteristics $\chi \in \mathfrak{a}$ and $\vartheta \in \mathfrak{b}$ such that $\chi(p) \leq \vartheta(p)$ for all $p \in \mathbb{P}$. The types $\mathfrak{a}$ and $\mathfrak{b}$ are said to be *comparable* if either $\mathfrak{a} \leq \mathfrak{b}$ or $\mathfrak{b} \leq \mathfrak{a}$. Otherwise, $\mathfrak{a}$ and $\mathfrak{b}$ are *incomparable*. If $\mathfrak{a}$, $\mathfrak{b}$ are types, then $\mathfrak{a} \wedge \mathfrak{b}$ is the type containing the characteristic

$$\langle \min\{\chi(p), \vartheta(p)\} \mid p \in \mathbb{P} \rangle,$$

where χ, ϑ are arbitrary elements of $\mathfrak{a}$, $\mathfrak{b}$ respectively.

We are now ready to begin the proof of Theorem 1.6. So let X be a standard Borel Γ-space with an invariant ergodic probability measure μ and let $f : X \to S(\mathbb{Q}^2)$ be a Borel function such that $x E_\Gamma^X y$ implies $f(x) \cong_2 f(y)$. Let $E = E_\Gamma^X$ and for each $x \in X$, let $G_x = f(x) \in S(\mathbb{Q}^2)$. Since μ is ergodic, there exists a Γ-invariant subset X_0 with $\mu(X_0) = 1$ such that one of the following two cases occurs.

(A) $G_x \in S_1(\mathbb{Q}^2)$ for all $x \in X_0$.
(B) $G_x \in S_2(\mathbb{Q}^2)$ for all $x \in X_0$.

To simplify notation, we shall assume that $X_0 = X$. (We shall make this simplifying assumption each time that we appeal to the ergodicity of μ.) First suppose that $G_x \in S_1(\mathbb{Q}^2)$ for all $x \in X$. By Theorem 1.3, $\cong_2 \restriction S_1(\mathbb{Q}^2)$ is hyperfinite; and so $\cong_2 \restriction S_1(\mathbb{Q}^2)$ is equal to $E_{\mathbb{Z}}^{S_1(\mathbb{Q}^2)}$ for some Borel action of $\mathbb{Z}$ on $S_1(\mathbb{Q}^2)$. By Theorem 2.4, there exists a Γ-invariant Borel subset M with $\mu(M) = 1$ such that f maps M into a single $\cong_2$-class. Hence we can suppose that $f : X \to S_2(\mathbb{Q}^2)$. Appealing to the ergodicity of μ once again, we can now suppose that one of the following three cases occurs.

(I) $|T(G_x)| > 2$ for all $x \in X$.
(II) $|T(G_x)| = 2$ for all $x \in X$.
(III) $|T(G_x)| = 1$ for all $x \in X$.

Case (I). Suppose that $|T(G_x)| > 2$ for all $x \in X$. By Król 2.5 [14], since each $|T(G_x)| > 2$, it follows that each $T(G_x)$ contains at least one pair of incomparable types. Let $g : X \to \mathbb{Q}^2$ be a Borel function which selects a basis $g(x) = \langle a_x, b_x \rangle \in G_x \times G_x$ such that the types $\mathfrak{a}_x = \tau(a_x)$ and $\mathfrak{b}_x = \tau(b_x)$ are incomparable. Let A_x, B_x be the pure subgroups of G_x generated by a_x, b_x respectively and let χ_x be the characteristic of the torsion group $G_x/(A_x + B_x)$. Suppose that $x \in X$ and that χ_x belongs to the zero-type. Then by Król 4.2 [14], it follows that $T(G_x) = \{\mathfrak{a}_x, \mathfrak{b}_x, \mathfrak{a}_x \wedge \mathfrak{b}_x\}$. In particular, $\{\mathfrak{a}_x, \mathfrak{b}_x\}$ is the *unique* pair of incomparable types in $T(G_x)$. Hence Lemma 3.1 implies that if xEy, then χ_x belongs to the zero-type iff χ_y belongs to the zero-type. Since μ is ergodic, we can suppose that one of the following two cases occurs.

(a) For all $x \in X$, χ_x belongs to the zero-type.
(b) For all $x \in X$, χ_x does not belong to the zero-type.

First suppose that χ_x belongs to the zero-type for all $x \in X$. Let

$$Y = (\mathbb{N} \cup \{\infty\})^{\mathbb{P}} \times (\mathbb{N} \cup \{\infty\})^{\mathbb{P}}$$

and let $h : X \to Y$ be the Borel function defined by $h(x) = \langle \chi(a_x), \chi(b_x) \rangle$. Let F be the countable Borel equivalence relation on Y such that

$$\langle \chi(a), \chi(b) \rangle \; F \; \langle \chi(c), \chi(d) \rangle \quad \text{iff} \quad \{\tau(a), \tau(b)\} = \{\tau(c), \tau(d)\}.$$

Since the similarity relation $\equiv$ on $(\mathbb{N} \cup \{\infty\})^{\mathbb{P}}$ is hyperfinite, it follows easily that there exists a Borel action of the amenable group $H = (\mathbb{Z} \times \mathbb{Z}) \rtimes \mathrm{Sym}(2)$ on Y such that $F = E_H^Y$; and clearly if xEy, then $h(x)Fh(y)$. Hence Theorem 2.4 implies that there exists a Γ-invariant subset X_0 with $\mu(X_0) = 1$ and a fixed pair of types $\{\mathfrak{a}, \mathfrak{b}\}$ such that $\{\mathfrak{a}_x, \mathfrak{b}_x\} = \{\mathfrak{a}, \mathfrak{b}\}$ for all $x \in X_0$. By Lemma 3.2, the image $f[X_0]$ consists of only countably many groups $G \in S_2(\mathbb{Q}^2)$. Hence there exists a Borel subset X_1 with $\mu(X_1) > 0$ and a fixed group $G \in S_2(\mathbb{Q}^2)$ such that $f(x) = G$ for all $x \in X_1$. Let $M = \Gamma.X_1$. Since μ is ergodic, $\mu(M) = 1$ and clearly $f[M]$ is contained in a single $\cong_2$-class.

Hence we can suppose that for all $x \in X$, χ_x does not belong to the zero-type. Let $g : X \to \mathcal{P}(\mathbb{P})$ be the Borel map defined by

$$g(x) = \{p \in \mathbb{P} \mid p^m G_x = G_x \text{ for all } m \geq 1\}.$$

Clearly if xEy, then $g(x) = g(y)$. By the ergodicity of μ, we can suppose that there exists a fixed set P of primes such that $g(x) = P$ for all $x \in X$. Let R be the subgroup of the multiplicative group $\mathbb{Q}^*$ consisting of those rational numbers of the form

$$r = \pm p_1^{m_1} p_2^{m_2} \dots p_k^{m_k},$$

where $p_1, \dots, p_k \in P$ and $m_1, \dots, m_k \in \mathbb{Z}$; and let

$$D = \{ \left(\begin{smallmatrix} r & 0 \\ 0 & r \end{smallmatrix}\right) \mid r \in R \} \leqslant GL_2(\mathbb{Q}).$$

(If $P = \emptyset$, then we set $R = \{1, -1\}$.) By Król 4.3 [14], $\mathrm{Aut}(G_x) = D$ for all $x \in X$. Let

$$Y = \{ G \in S_2(\mathbb{Q}^2) \mid \text{ There exists } x \in X \text{ such that } G_x \cong_2 G \}.$$

Then Y is a $GL_2(\mathbb{Q})$-invariant Borel subset of $S_2(\mathbb{Q}^2)$; and the action of $GL_2(\mathbb{Q})$ on Y induces a free action of $H = GL_2(\mathbb{Q})/D$ on Y. Hence we can define a Borel cocycle $\beta : \Gamma \times X \to H$ by

$$\begin{aligned} \beta(\pi, x) &= \text{ the unique } \varphi \in H \text{ such that } \varphi[G_x] = G_{\pi.x} \\ &= \text{ the unique } \varphi \in H \text{ such that } \varphi[f(x)] = f(\pi.x) \end{aligned}$$

Let Z be the center of $GL_2(\mathbb{Q})$ and let $p : H \to PGL_2(\mathbb{Q}) = GL_2(\mathbb{Q})/Z$ be the natural surjective homomorphism. Then we can define a Borel cocycle

$$\alpha : \Gamma \times X \to PGL_2(\mathbb{Q})$$

by $\alpha = p \circ \beta$. By Theorem 2.3, there exists an equivalent cocycle

$$\gamma : \Gamma \times X \to PGL_2(\mathbb{Q})$$

such that $\gamma[\Gamma \times X]$ is contained in a finite subgroup K of $PGL_2(\mathbb{Q})$. Let $B : X \to PGL_2(\mathbb{Q})$ be a Borel function such that:

$$(*) \qquad \text{for all } \pi \in \Gamma,\ \alpha(\pi, x) = B(\pi.x)\gamma(\pi, x)B(x)^{-1} \quad \mu\text{-a.e.}$$

It is easily checked that if x satisfies (*) and xEy, then y also satisfies (*). To simplify notation, we shall assume that (*) holds for all $x \in X$. Clearly there exists a Borel subset $X_1 \subseteq X$ with $\mu(X_1) > 0$ and a fixed element $\psi \in PGL_2(\mathbb{Q})$ such that $B(x) = \psi$ for all $x \in X_1$. Since μ is ergodic, $\mu(\Gamma.X_1) = 1$; and so we can also assume that X_1 intersects every Γ-orbit on X.

Let $\{\pi_n \mid n \in \mathbb{N}\}$ be a fixed enumeration of Γ with $\pi_0 = 1$; and for each $x \in X$, let $x_1 = \pi_n.x$, where n is the least integer such that $\pi_n.x \in X_1$. Notice that for each $x \in X$,

$$\Psi(x) = \{\alpha(\pi, x_1) \mid \pi.x_1 \in X_1\} \subseteq \psi K \psi^{-1}$$

is a finite subset of $PGL_2(\mathbb{Q})$. Let $L = Z/D$ so that $PGL_2(\mathbb{Q}) = H/L$; and regard $\alpha(\pi, x_1)$ as a coset of L in H. Then $\beta(\pi, x_1) \in \alpha(\pi, x_1)$. It follows that

$$\Phi(x) = \{L.f(\pi.x_1) \mid \pi.x_1 \in X_1\}$$

is also finite of cardinality at most $|K|$; and it is easily checked that if xEy, then $\Phi(x) = \Phi(y)$. By the ergodicity of μ, we can suppose that there exists an integer $1 \leq k \leq |K|$ such that $|\Phi(x)| = k$ for all $x \in X$. For each $x \in X$ and $1 \leq i \leq k-1$, let $x_{i+1} = \pi_n.x_1$, where n is the least integer such that

(i) $\pi_n.x_1 \in X_1$, and

(ii) $f(\pi_n.x_1) \notin \{L.f(x_j) \mid 1 \leq j \leq i\}$.

Let $\widetilde{f} : X \to S_2(\mathbb{Q}^2)^k$ be the Borel function defined by

$$\widetilde{f}(x) = \langle f(x_1), \ldots, f(x_k)\rangle.$$

Let F be the countable Borel equivalence relation arising from the orbits of the natural action of

$$W = \underbrace{(L \times \cdots \times L)}_{k \text{ copies}} \rtimes \mathrm{Sym}(k)$$

on $S_2(\mathbb{Q}^2)^k$. If xEy, then $\Phi(x) = \Phi(y)$ and so $\widetilde{f}(x) F \widetilde{f}(y)$. Clearly W is an amenable group; and so by Theorem 2.4, there exists a Γ-invariant Borel subset M with $\mu(M) = 1$ such that $\widetilde{f}$ maps M into a single F-class; and this implies that f maps M into a single $\cong_2$-class.

Case (II). Suppose that $|T(G_x)| = 2$ for all $x \in X$. By Król 2.6 [14], the types $\mathfrak{a}_x$, $\mathfrak{b}_x \in T(G_x)$ are comparable; say, $\mathfrak{a}_x < \mathfrak{b}_x$. Furthermore, by Król 2.26 [14], if b_1, $b_2 \in G_x$ satisfy $\tau(b_1) = \tau(b_2) = \mathfrak{b}_x$, then b_1 and b_2 are linearly dependent. It follows that there exists a Borel function $g : X \to GL_2(\mathbb{Q})$ such that

$$g(x)[\{b \in G_x \mid \tau(b) = \mathfrak{b}_x\}] \leqslant \{\left(\begin{smallmatrix} q \\ 0 \end{smallmatrix}\right) \mid q \in \mathbb{Q}\}$$

for all $x \in X$. Let $f' : X \to S_2(\mathbb{Q}^2)$ be the Borel function defined by $f'(x) = g(x)[G_x]$ and let $G'_x = g(x)[G_x]$. Let H be the subgroup consisting of the upper triangular matrices of $GL_2(\mathbb{Q})$. Then H is soluble and hence is amenable. Notice that if $\varphi \in GL_2(\mathbb{Q})$ satisfies $\varphi[G'_x] = G'_y$, then

$$\varphi[\{b \in G'_x \mid \tau(b) = \mathfrak{b}_x\}] = \{c \in G'_y \mid \tau(c) = \mathfrak{b}_y\};$$

and so φ has the form

$$\varphi = \begin{pmatrix} r & s \\ 0 & t \end{pmatrix}$$

for some r, s, $t \in \mathbb{Q}$. Thus $G'_x \cong_2 G'_y$ iff there exists $\varphi \in H$ such that $\varphi[G'_x] = G'_y$. By Theorem 2.4, there exists a Γ-invariant Borel subset M with $\mu(M) = 1$ such that f' maps M into a single H-orbit on $S_2(\mathbb{Q}^2)$; and this implies that f maps M into a single $\cong_2$-class.

Case (III). Suppose that $|T(G_x)| = 1$ for all $x \in X$; say, $T(G_x) = \{\mathfrak{a}_x\}$. Arguing as in Case (I), we can suppose that there exists a fixed type $\mathfrak{a}$ such that $\mathfrak{a}_x = \mathfrak{a}$ for all $x \in X$. After slightly adjusting f if necessary, we can also suppose that for all $x \in X$, the standard basis elements e_1, e_2 of $\mathbb{Q}^2$ are contained in G_x and that e_1, e_2 realise the same characteristic in G_x. (With a little more effort, we could even reduce to the case when the characteristic realised by e_1, e_2 in G_x is fixed for all $x \in X$. However, this extra uniformity is not required in the following argument.) Let A_x, B_x be the pure subgroups of G_x generated by e_1, e_2 respectively and let χ_x be the characteristic of the torsion group $G_x/(A_x + B_x)$. Arguing as in Case (I), we can also suppose that for all $x \in X$, χ_x does not belong to the zero-type.

Since $\mathfrak{a}_x = \mathfrak{a}$ for all $x \in X$, it follows that there is a fixed set P of primes such that

$$P = \{p \in \mathbb{P} \mid p^m G_x = G_x \text{ for all } m \geq 1\}$$

for all $x \in X$. Let R be the subgroup of the multiplicative group $\mathbb{Q}^*$ consisting of those rational numbers of the form

$$r = \pm p_1^{m_1} p_2^{m_2} \dots p_k^{m_k},$$

where $p_1, \dots, p_k \in P$ and $m_1, \dots, m_k \in \mathbb{Z}$; and let

$$D = \{\left(\begin{smallmatrix} r & 0 \\ 0 & r \end{smallmatrix}\right) \mid r \in R\} \leqslant GL_2(\mathbb{Q}).$$

(Once again, if $P = \emptyset$, then we set $R = \{1, -1\}$.) Then $D \leqslant \mathrm{Aut}(G_x)$ for all $x \in X$. By the ergodicity of μ, we can suppose that one of the following two cases occurs.

(a) $D = \mathrm{Aut}(G_x)$ for all $x \in X$.

(b) D is a proper subgroup of $\mathrm{Aut}(G_x)$ for all $x \in X$.

First suppose that $D = \mathrm{Aut}(G_x)$ for all $x \in X$; and let

$$Y = \{G \in S_2(\mathbb{Q}^2) \mid \text{ There exists } x \in X \text{ such that } G_x \cong_2 G\}.$$

Then the action of $GL_2(\mathbb{Q})$ on Y induces a free action of $H = GL_2(\mathbb{Q})/D$ on Y; and hence we can define a Borel cocycle $\beta : \Gamma \times X \to H$ by

$$\beta(\pi, x) = \text{ the unique } \varphi \in H \text{ such that } \varphi[G_x] = G_{\pi.x}.$$

Arguing as in Case (I), we see that there exists a Γ-invariant Borel subset M with $\mu(M) = 1$ such that f maps M into a single $\cong_2$-class.

Hence we can suppose that D is a proper subgroup of $\mathrm{Aut}(G_x)$ for all $x \in X$. Fix some $x \in X$ and let

$$\varphi = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{Aut}(G_x) \smallsetminus D.$$

By Król 4.8 [14], $bc \neq 0$ and $\sqrt{d} \notin \mathbb{Q}$, where

$$d = (a + d)^2 - 4(ad - bc).$$

Let $\mathrm{QE}(G_x)$ be the ring of quasi-endomorphisms of G_x; i.e. $\mathrm{QE}(G_x)$ consists of those linear transformations $\eta \in \mathrm{Mat}_2(\mathbb{Q})$ such that there exists an integer $m \neq 0$ with $m\eta \in \mathrm{End}(G_x)$, the endomorphism ring of G_x. By Król 2.29 [14], since the characteristic χ_x of the torsion group $G_x/(A_x + B_x)$ does not belong to the zero-type, it follows that G_x is not decomposable into a direct sum of rank one subgroups. Hence by Król 5.10 [14], $\mathrm{QE}(G_x)$ is the ring S_x of all matrices $\eta \in \mathrm{Mat}_2(\mathbb{Q})$ such that there exists a rational number $q \in \mathbb{Q}$ and an arbitrary scalar matrix λ such that

$$\eta = q\varphi + \lambda.$$

Furthermore, by Król 5.11 [14], S_x is isomorphic to the quadratic field extension $\mathbb{Q}(\sqrt{d})$ via the map which sends the identity matrix to $1 \in \mathbb{Q}$ and sends φ to $\sqrt{d}$.

In particular, there are only countably many possibilities for $QE(G_x)$; and so there exists a Borel subset X_1 with $\mu(X_1) > 0$ and a fixed subring S of $\mathrm{Mat}_2(\mathbb{Q})$ such that $QE(G_x) = S$ for all $x \in X_1$. By the ergodicity of μ, we can suppose that $X = \Gamma.X_1$; and after slightly adjusting f if necessary, we can suppose that $QE(G_x) = S$ for all $x \in X$. Thus if x, $y \in X$, then $G_x \cong_2 G_y$ iff there exists $\psi \in N = N_{GL_2(\mathbb{Q})}(S)$ such that $\psi[G_x] = G_y$. Hence by Theorem 2.4, it is enough to show that N is an amenable group. To see this, let $S \cong \mathbb{Q}(\sqrt{d})$, where $\sqrt{d} \notin \mathbb{Q}$. Since $\mathrm{Aut}(S) \cong \mathrm{Aut}(\mathbb{Q}(\sqrt{d}))$ is cyclic of order 2, it follows that the centraliser $C = C_N(S)$ satisfies $[N : C] \leq 2$. By the Double Centraliser Theorem for finite dimensional central simple algebras,

$$\dim_{\mathbb{Q}} S \ . \ \dim_{\mathbb{Q}} C_{\mathrm{Mat}_2(\mathbb{Q})}(S) = \dim_{\mathbb{Q}} \mathrm{Mat}_2(\mathbb{Q}).$$

(For example, see Section 12.7 [18].) It follows that $C_{\mathrm{Mat}_2(\mathbb{Q})}(S) = S$ and hence C is the multiplicative group of the field S. Consequently N is abelian-by-finite and so N is amenable.

This completes the proof of Theorem 1.6.

4. A cocycle reduction result

In this section, we shall prove Theorem 2.3. But first we need to recall some notions from valuation theory. (A clear account of this material can be found in Margulis [17, Chapter I].) Let F be an algebraic number field; i.e. a finite extension of the field $\mathbb{Q}$ of rational numbers. Let $\mathcal{R}$ be the set of all non-equivalent valuations of F and let $\mathcal{R}_\infty \subset \mathcal{R}$ be the set of archimedean valuations. For each $\nu \in \mathcal{R}$, let F_ν be the completion of F relative to ν. If $\nu \in \mathcal{R}_\infty$, then $F_\nu = \mathbb{R}$ or $F_\nu = \mathbb{C}$; and if $\nu \in \mathcal{R} \smallsetminus \mathcal{R}_\infty$, then F_ν is a totally disconnected local field; i.e. a finite extension of the field $\mathbb{Q}_p$ of p-adic numbers for some prime p.

Let $S \subseteq \mathcal{R}$ be a set of valuations of F. Then an element $x \in F$ is said to be *S-integral* iff $|x|_\nu \leq 1$ for each non-archimedean valuation $\nu \notin S$. The set of all S-integral elements is a subring of F, which will be denoted by $F(S)$. Furthermore, F is the union of the subrings $F(S)$, where S ranges over the *finite* sets of valuations of the field F. For any $S \subseteq \mathcal{R}$, we define

$$GL_n(F(S)) = \{\varphi \in GL_n(F) \mid \text{ The entries of } \varphi \text{ and } \varphi^{-1} \text{ belong to } F(S)\};$$

and for each F-subgroup $H(F)$ of $GL_n(F)$, we define $H(F(S)) = H(F) \cap GL_n(F(S))$.

Theorem 2.3 is a straightforward consequence of the following theorem, which collects together results of Zimmer [25] and Adams-Spatzier [2].

Theorem 4.1. *Let Γ be a countable Kazhdan group and let X be a standard Borel Γ-space with an invariant ergodic probability measure μ. Suppose that $\alpha : \Gamma \times X \to PSL_2(K)$ is a Borel cocycle, where either $K = \mathbb{R}$ or K is a totally disconnected local field. Then there exists an equivalent cocycle γ such that $\gamma[\Gamma \times X]$ is contained in a compact subgroup of $PSL_2(K)$.*

Proof. First suppose that $K = \mathbb{R}$. By Zimmer [25, Theorem 10], α is equivalent to a cocycle γ such that

$$\gamma[\Gamma \times X] \subseteq H \leqslant PSL_2(\mathbb{R}),$$

where H is an algebraic subgroup of $PSL_2(\mathbb{R})$ such that H is a Kazhdan group; and Corollary 19 [25] implies that every Kazhdan subgroup of $PSL_2(\mathbb{R})$ is compact.

Now suppose that K is a totally disconnected local field. By Serre [20], there is an action of $PSL_2(K)$ as a group of automorphisms of a countable simplicial tree T such that the stabilizer of each point of T is compact; and by Adams-Spatzier [2, Theorem 1.1], α is equivalent to a cocycle γ such that $\gamma[\Gamma \times X]$ is contained in the stabilizer of some point of T. □

We are now ready to begin the proof of Theorem 2.3. So let Γ be a countable Kazhdan group and let X be a standard Borel Γ-space with an invariant ergodic probability measure μ. Suppose that $\alpha : \Gamma \times X \to PGL_2(\mathbb{Q})$ is a Borel cocycle. After passing to a finite ergodic extension of X if necessary, we can suppose that $\alpha[\Gamma \times X] \subseteq PGL_2^+(\mathbb{Q})$, the subgroup of $PGL_2(\mathbb{Q})$ of index 2 corresponding to the matrices $\varphi \in GL_2(\mathbb{Q})$ such that $\det(\varphi) > 0$. (For example, see Propositions 2.5 and 2.6 [1].) Since Γ is a Kazhdan group, Lemma 2.2 [26] implies that α is equivalent to a cocycle β taking values in a finitely generated subgroup Λ of $PGL_2^+(\mathbb{Q})$. So there exists a finite set $\{p_1, \dots, p_n\}$ of primes and a finite set S of valuations of the algebraic number field $F = \mathbb{Q}[\sqrt{p_1}, \dots, \sqrt{p_n}]$ such that $\Lambda \leqslant PSL_2(F(S))$. Clearly we can suppose that S contains the set $\mathcal{R}_\infty$ of archimedean valuations of F. It follows that if $PSL_2(F(S))$ is identified with its image under the diagonal embedding into

$$G_S = \prod_{\nu \in S} PSL_2(F_\nu),$$

then $PSL_2(F(S))$ is a discrete subgroup of G_S. (For example, see Section I.3.2 [17].) Of course, F_ν is a totally disconnected local field for each $\nu \in S \smallsetminus \mathcal{R}_\infty$; and since F is a totally real field, it follows that $F_\nu = \mathbb{R}$ for all $\nu \in \mathcal{R}_\infty$. For each $\nu \in S$, let $p_\nu : G_S \to PSL_2(F_\nu)$ be the canonical projection; and, viewing β as a cocycle into G_S, let $\beta_\nu : \Gamma \times X \to PSL_2(F_\nu)$ be the Borel cocycle defined by $\beta_\nu = p_\nu \circ \beta$. By Theorem 4.1, β_ν is equivalent to a cocycle taking values in a compact subgroup H_ν of $PSL_2(F_\nu)$. It follows that β is equivalent to a cocycle taking values in the compact subgroup $H = \prod_{\nu \in S} H_\nu$ of G_S. By Adams-Kechris [1, Proposition 2.4], there exists $g \in G_S$ and a cocycle $\gamma : \Gamma \times X \to \Lambda$ such that $\beta \sim \gamma$ and γ takes values in the finite subgroup $\Lambda \cap gHg^{-1}$ of $PGL_2(\mathbb{Q})$. This completes the proof of Theorem 2.3.

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