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P. LARSON and S. SHELAH

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INSTITUT MITTAG-LEFFLER
THE ROYAL SWEDISH ACADEMY OF SCIENCES

Bounding by canonical functions, with CH[*†]

Paul Larson Saharon Shelah[‡]

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Abstract

We show that that a certain class of semi-proper iterations does not add ω-sequences. As a result, starting from suitable large cardinals one can obtain a model in which the Continuum Hypothesis holds and every function from ω_1 to ω_1 is bounded by a canonical function on a club, and so ω_1 is the ω_2-nd canonical function.

1 Introduction

Given an ordinal γ, a function $f : \omega_1 \to Ord$ is a canonical function for γ if the empty condition (i.e., ω_1) in the forcing $\mathcal{P}(\omega_1)/I_{NS}$ forces that $j(f)(\omega_1^V) = \gamma$, where j is the elementary embedding induced by the generic. For each $\alpha < \omega_1$, the constant function with value α is the canonical function for α. For $\alpha \in [\omega_1, \omega_2)$, a canonical function f for α is obtained by taking a bijection $g : \omega_1 \to \alpha$ and letting $f(\beta) = o.t.(g[\beta])$. The first interesting questions about canonical functions are whether there is a canonical function for ω_2, and if there is, whether it is the constant function ω_1. If the nonstationary ideal on ω_1 (I_{NS}) is saturated, then every function from ω_1 to ω_1 is bounded on a club subset of ω_1 by a canonical function for an ordinal less than ω_2. This implies that ω_1 is the ω_2-nd canonical function.

The most quotable result in this paper is that the standard forcing to make every function from ω_1 to ω_1 bounded by a canonical function does not add ω-sequences, and so this statement is consistent with the Continuum Hypothesis. We give a more general theorem stating that a certain class of semi-proper iterations does not add ω-sequences. This class includes the standard forcing to make ω_1 the ω_2-nd function, and is general enough to show that a generalization of bounding for sets of reals is also consistent with CH, answering a question in [6].

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One of the key constructions used in the proof generalizes the notion of α-properness from Chapter V of [3] to semi-proper forcing. Briefly, a forcing is α-semi-proper if for any $\in$-chain of countable elementary submodels of length α, there is a condition which is simultaneously semi-generic for each model in the sequence. The problem in applying the method to show that a given Revised Countable Support iteration does not add ω-sequences is that for a given model N in the sequence, $N[G_{p_\alpha}] \cap X \cap length(\bar{Q})$ can increase, so that new steps in the iteration appear. For the forcings in this paper, however, we have a good understanding of how to enlarge each such N, as well as how to produce the appropriate tower of models to overcome this.

This can be generalized further, getting the consistency of certain forcing axioms, using ideas from [3], Chapters V and VIII, and [5]. The reader is referred to [2] for more on this topic and on RCS in particular.

Woodin [6] has shown that if I_{NS} is saturated and large cardinals exist, then CH fails. One motivation for the work in this paper was to see whether this theorem holds with the weaker hypothesis of bounding by canonical functions. We have shown that it doesn't. Another reason for interest in this question derives from the study of Woodin's $\mathbb{P}_{max}$ forcing, which he uses to produce a model in which CH fails and all consistent Π_2 sentences for $H(\omega_2)$ hold simultaneously. It is not known whether all such Π_2 sentences consistent with CH can hold together with CH. The generalized form of bounding in this paper is a candidate for showing that this is impossible. Candidates for the other half of the incompatibility appear in [4].

2 A scheme for not adding ω-sequences

The following theorem is the main theorem of the paper.

Theorem 2.1 *Let $\langle \lambda_\alpha : \alpha < \kappa \rangle$ be a continuous increasing sequence of ordinals with supremum κ strongly inaccessible. Fix a regular cardinal $\chi > (2^\kappa)^+$, and let $<_\chi$ be a wellordering of $H(\chi)$. Let $\langle A^\alpha_\beta \subset [\lambda_{\alpha+1}]^{<\omega_1} : \alpha < \kappa, \beta < \omega_1 \rangle$ be such that*

1. *for all $\alpha < \kappa, \beta < \omega_1, E \in A^\alpha_\beta$, if $E \subset E'$ and $E \cap \omega_1 = E' \cap \omega_1$ then $E' \in A^\alpha_\beta$,*
2. *there exists a club C of countable $X \prec (H(\chi), \in, <_\chi)$ with the property that for any $\alpha \in X \cap \kappa, \beta < \omega_1$ there exists $Y \prec (H(\chi), \in, <_\chi)$ such that*
   - *(a) $X \subset Y$ and $Y \in C$,*
   - *(b) $X \cap V_{\lambda_\alpha+1} = Y \cap V_{\lambda_\alpha+1}$,*
   - *(c) $Y \cap \lambda_{\alpha+1} \in A^\alpha_\beta$.*

For each $\alpha < \kappa, f \in \omega_1^{\omega_1}$ let $Q_{\alpha,f}$ be the forcing whose conditions are countable, continuous, increasing sequences $\langle x_\beta : \beta \leq \gamma \rangle$ such that for each $\beta \leq \gamma$, $x_\beta \cap \omega_1 \in \omega_1$ and $x_\beta \in A^\alpha_{f(x_\beta \cap \omega_1)}$.

Let $\bar{Q} = \langle P_\alpha, \underset{\sim}{Q}_\alpha, \underset{\sim}{f}_\alpha : \alpha < \kappa \rangle$ be such that *(a)* $\bar{P} = \langle P_\alpha, \underset{\sim}{Q}_\alpha : \alpha < \kappa \rangle$ *is a Revised Countable Support iteration,*

(b) each P_α *has cardinality (or, formally, density)* $\leq \lambda_\alpha$ *for nonlimit* α, *and* $\leq 2^{\lambda_\alpha}$ *for* α *a successor.*

(c) $\underset{\sim}{f}_\alpha$ *is a* P_β*-name for a function from* ω_1 *to* ω_1, *for* $\beta = 0$ *or* $\beta < \alpha$,

(d) for all $\alpha < \kappa$, $1_{P_\alpha} \Vdash \underset{\sim}{Q}_\alpha = Q_{\alpha, \underset{\sim}{f}_\alpha}$.

Then forcing with $\bar{P}$ *does not add* ω*-sequences.*

Since in the end we show that $\bar{P}$ does not add ω-sequences, our iteration is actually by Countable Support, and so the whole proof can be rearranged without mention of RCS. We have defined it this way for expositional convenience.

One could strengthen the theorem by relaxing the upwards closure of the A^α_β, requiring that Condition 1 above holds only for some $\eta < \lambda_\alpha$, instead of for ω_1. Notice that Conditon 2b implies in most cases that each $\lambda_{\alpha+1} > 2^{\lambda_\alpha}$.

For the rest of this paper, sets denoted by κ, $\langle \lambda_\alpha : \alpha < \kappa \rangle$, χ, $<_\chi$, C, $Q_{\alpha,f}$, $\bar{Q}$, $\bar{P}$ and $\langle A^\alpha_\beta : \alpha < \kappa, \beta < \omega_1 \rangle$ are supposed to have the properties given in the hypotheses of Theorem 2.1. This policy will be modified in two ways. First, in the application sections, we will add extra properties for this terms. On the other hand, in proving the main theorem we sometimes state our lemmas more generally in terms of certain properties of these objects, temporarily forgetting the others. We hope that it is clear when we are doing this, so that there will be no confusion.

3 Applications

3.1 Bounding

We let Bounding denote the statement that every function from ω_1 to ω_1 is bounded on a club by a canonical function. In this section each A^α_β is the set of countable subsets of $\lambda_{\alpha+1}$ of ordertype greater than β. Then for a function $f : \omega_1 \to \omega_1$ and $\alpha < \kappa$, $Q_{\alpha,f}$ is the forcing whose conditions are countable, continuous, increasing sequences $\langle a_\xi \in [\lambda_{\alpha+1}]^{<\omega_1} : \xi \leq \gamma \rangle$ such that for each $\xi \leq \gamma$, $a_\xi \cap \omega_1 \in \omega_1$ and $o.t.(a_\xi) > f(a_\xi \cap \omega_1)$, ordered by extension. $Q_{\alpha,f}$ makes $\lambda_{\alpha+1}$ have cardinality $\aleph_1$, and makes the canonical function for $\lambda_{\alpha+1}$ dominate f on a club.

Definition 3.1 *A cardinal* λ *is called end-extending if for all* $\chi > (2^\lambda)^+$, *and* $N \prec (H(\chi), \in, <^*_\chi)$ *countable,* $\lambda \in N$, *and all* $\gamma < \omega_1$, $\eta < \lambda$ *there is a countable* $M \prec (H(\chi), \in, <^*_\chi)$ *containing* N *such that* $M \cap \eta = N \cap \eta$ *and* $o.t.(M \cap \lambda) > \gamma$.

It is a standard fact that measurable cardinals are end-extending.

Corollary 3.2 *Let* κ *be strongly inaccessible and let* $\langle \lambda_\alpha : \alpha < \kappa \rangle$ *be a continuous, increasing sequence of ordinals with supremum* κ *such that each* $\lambda_{\alpha+1}$

is end-extending. Let $\langle A^\alpha_\beta : \alpha < \kappa, \beta < \omega \rangle$ be such that each A^α_β is the set of countable subsets of $\lambda_{\alpha+1}$ of ordertype greater than β. Then these sets satisfy the conditions of Theorem 2.1. Let $\bar{P}$ be a forcing iteration as in Theorem 2.1, with the additional stipulation that if $\underset{\sim}{f}$ is a P_α-name for a function from ω_1 to ω_1, then for some $\beta \in [\alpha, \kappa)$, $\underset{\sim}{f_\beta} = \underset{\sim}{f}$, Then after forcing with $\bar{P}$ as defined in that theorem we have that the Continuum Hypothesis holds and every function from ω_1 to ω_1 is bounded by a canonical function on a club.

3.2 Souslin Bounding

The previous application can be generalized to show that the following statement can be forced without adding reals, answering a question in [6].

Definition 3.3 *Souslin Bounding is the following statement. Say that $A \subset \mathbb{R}$ is universally Baire, and that $f : \omega_1 \to A$. Then there is a tree T on $\omega \times \omega_1$ such that $A = p[T]$ and such that $\{\alpha < \omega_1 \mid f(\alpha) \in p[T \restriction (\omega \times \alpha)]\}$ contains a club.*

Theorem 3.4 *Bounding is equivalent to Souslin Bounding for $\underset{\sim}{\Pi}^1_1$ sets.*

Proof: For the forward direction, note that it suffices to consider the case where A is the set W of reals coding countable well orderings. Given $f : \omega_1 \to W$, let $\beta < \omega_2$ be such that each canonical function for β dominates the function f' on a club, where $f'(\alpha)$ is the length of the well ordering coded by $f(\alpha)$. Fix a bijection $g : \omega_1 \to \beta$, and let T be the set of $\langle (n_i, \eta_i) \in \omega \times \omega_1 : i < m \rangle$ such that the n_i's code a linear ordering on ω, and the η_i's have the property that in the ordering $<^*$ coded by the n_i's, $i <^* j$ if and only if $g(\eta_i) < g(\eta_j)$. Then the projection of T is exactly W, and each $f(\alpha) \in p[T \restriction (\omega \times \alpha)]$ since the ordertype of $g[\alpha]$ is greater than $f(\alpha)$.

For the other direction, fix $f : \omega_1 \to \omega_1$ and again let W be a the set of function g from ω to ω coding wellorderings of ω, with the additional stipulation that $g(n)$ codes how n compares with each $m < n$ in the corresponding ordering. Consider f as a function from ω_1 into W, and let T be the tree given by Souslin Bounding. We have a partial order $<_o$ on the sequences in T, $\sigma_0 <_o \sigma_1$ if one extends the other, and, σ_i being the longer one, the first cordinate of the last element of σ_i codes that $|\sigma_0| < |\sigma_1|$ in the corresponding ordering. Since the ordering corresponding to any path through T is wellfounded, this partial order is also wellfounded, and so we can extend it to a wellordering. Now any enumeration of T of length ω_1 induces a canonical function on ω_1 by the this wellordering, and by the property of T with respect to f given by Souslin Bounding, this canonical function will be at least as large as f on a club. Since we can do this for any f, we are done. ■

Definition 3.5 *([1]) Suppose that $A \subset \mathbb{R}$. Then A is universally Baire if for any compact Hausdorff space X and any continuous function $\pi : X \to \mathbb{R}$ the set $\{a \in X \mid \pi(a) \in A\}$ has the property of Baire in X.*

The context and characterization of universal Baireness which we will be considering are given by Theorem 3.7. Given a set X, we let $m(X)$ denote the set of measures on X. The reader is referred to [6] for more detail.

Definition 3.6 *Suppose that κ is a nonzero ordinal and that T is a tree on $\omega \times \kappa$. Then T is δ-homogeneous if there is a partial function $\pi : \omega^{<\omega} \to m(\kappa^{<\omega})$ such that*

1. *if $s \in dom(\pi)$ then $\pi(s)(T_s) = 1$ and $\pi(s)$ is a δ-complete measure.*
2. *for all $x \in \omega^\omega$, $x \in p[T]$ if and only if*
   (a) *$\{x \restriction k : k \in \omega\} \subset dom(\pi)$,*
   (b) *$\langle \pi(x \restriction k) : k \in \omega \rangle$ is a countably complete tower.*

A set $A \subset \mathbb{R}$ is δ-homogeneously Souslin if $A = p[T]$ for some δ-homogeneous tree T. A is $^\infty$-homogeneously Souslin if it is δ-homogeneously Souslin for arbitrarily large δ.

Theorem 3.7 *([1]) Suppose that there is a proper class of Woodin cardinals and that $A \subset \mathbb{R}$. Then the following are equivalent.*

1. *A is universally Baire.*
2. *A is $^\infty$-homogeneously Souslin.*

We use the following fact to ensure that our forcing iteration considers all universally Baire sets.

Lemma 3.8 *Let P be a forcing which does not add ω-sequences, and let $G \subset P$ be V-generic. Then in $V[G]$, for all $A \subset \mathbb{R}$ and all $\gamma > (2^{|P|})^+$, A is γ-homogeneously Souslin if and only if $A \in V$ and A is γ-homogeneously Souslin in V.*

Proof: This follows from the following standard facts about measures, where $V[G]$ is an extension by a forcing P such that $(2^{|P|})^+ < \gamma$.

1. For every γ-complete measure U on $\kappa^{<\omega}$ in $V[G]$, $U \cap V \in V$.
2. Every γ-complete measure on $\kappa^{<\omega}$ in V extends to one in $V[G]$.
3. For every γ-complete measure on $\kappa^{<\omega}$ in $V[G]$, every positive set contains a positive set in V.

For the forward direction, let T be a γ-homogeneous tree on $\omega \times \kappa$ in $V[G]$ such that $p[T] = A$, for some $\delta > (2^{|P|})^+$. Let $\pi : \omega^{<\omega} \to m(\kappa^{<\omega})$ witness that T is γ-homogeneous. Each $\pi(\sigma)$ extends a measure in V on $\kappa^{<\omega}$, and since P added no ω-sequences, the corresponding function π' taking each σ to the restriction of $\pi(\sigma)$ to V exists in V. For each $x \in \mathbb{R} \setminus p[T]$, let $\langle A^x_k : k < \omega \rangle$ be a witness to the fact that $\langle \pi(x\restriction k) : k < \omega \rangle$ is not countably complete. For each $s \in \omega^{<\omega}$, let

$$B_s = \{t \in \lambda \mid t \in \bigcap \{A^x_{length(s)} \mid x \in \mathbb{R} \setminus p[T] \wedge s \subset x\}\}.$$

Since every positive set for each $\pi(x\restriction k)$ contains one from V, and since P adds no ω-sequences, we can assume by shrinking if necessary that $\langle B_s \mid s \in \omega^{<\omega}\rangle$ is in V. Now let $T' \in V$ be the set of pairs s, t such that $t \in B_s$. Since the measures are all γ-complete, each B_s is positive for $\pi(s)$, and T' is as desired.

For the other direction, assuming that T and π are in V, we extend the $\pi(\sigma)$'s to $V[G]$-measures, noting that each positive set in $V[G]$ must contain one in V. Then π' witnesses that T is γ-homogeneous in $V[G]$, and since no ω-sequences have been added the projection is the same. ■

Let T be a tree on $\omega \times \kappa$ and let $f : \omega_1 \to p[T]$. Our one-step forcing $R_{f,T}$ is the set of continuous increasing countable sequences $\langle x_\alpha \in [\kappa]^{<\omega_1} : \alpha \leq \beta\rangle$ of countable length such that for all $\alpha \leq \beta$, $x_\alpha \cap \omega_1 \in \omega_1$ and $f(x_\alpha \cap \omega_1) \in p[T\restriction(\omega \times x_\alpha)]$, ordered by extension.

Given a δ^+-homogeneous set of reals, we use the measures witnessing the homogeneity to suitably expand countable elementary substructures of $H(\chi)$.

Lemma 3.9 *For some $\delta \geq \omega_1$, let T be a δ^+-homogeneous tree on $\omega \times \kappa$ and let $f : \omega_1 \to p[T]$. Let χ be a large enough regular cardinal with $<_\chi$ a wellordering of $H(\chi)$, and let $X \prec H(\chi)$ with T and f in X. Then for any countable subset a of $p[T]$ there exists a countable set $z \subset \kappa$ such that letting $Y = Sk_{(H(\chi),\in,<_\chi)}(X \cup z)$ we have that*

1. *$X \cap \delta = Y \cap \delta$,*
2. *$a \subset p[T\restriction(\omega \times (Y \cap \kappa))]$.*

Proof: Fix T, f, X and $a = \{a_i : i < \omega\}$. It suffices to show that we can deal with a_0, as we can just repeat the process ω times. For each $k < \omega$, let $A_k = \bigcap(X \cap \pi(a_0\restriction k))$. Then by the definition of δ^+-homogeneous there exists $z \in (\kappa \setminus \delta)^\omega$ such that for all $k < \omega$, $(a_0\restriction k, z\restriction k) \in T$, and $z\restriction k \in A_k$. Now $Y = Sk_{(H(\chi),\in,<_\chi)}(X \cup z)$ is as desired, since by the δ^+-completeness of each $\pi(a_0\restriction k)$, if $f \in X$ and $f(a_0\restriction k) \in \delta$ then $f(a_0\restriction k) \in X$. ■

Given Y as in Lemma 3.9 where $f(Y \cap \omega_1) \in a$, the union of any Y-generic for $R_{f,T}$ is a condition.

To get the consistency of Souslin Bounding from Theorem 2.1, we start from a proper class of Woodin cardinals, and let κ be a strongly inaccessible cardinal such that every $^\infty$-homogeneously Souslin set of reals is $^\infty$-homogeneously Souslin in V_κ. The assumption of the Woodin cardinals is just to make $^\infty$-homogeneous Souslinity equal to universal Baireness. Let $F : \omega_1 \to \mathbb{R}$ be a wellordering of the reals. Let our bookkeeping for $\bar{P}$ be such that each pair (A, f), where A is a universally Baire set and $f : \omega_1 \to A$ is added by some initial segment of the iteration, is associated to some α greater than the stage at which f was added, such that A is the projection of a $(2^{\lambda_\alpha})^+$-homogeneously Souslin tree $T \in V_\kappa$ on $\omega \times \delta$ for some $\delta \leq \lambda_{\alpha+1}$. For this α, we let each A^α_β be the set of countable $x \subset \lambda_{\alpha+1}$ such that $F(\beta) \in p[T]\restriction(\omega \times x)] \cup \bar{A}$, and so $Q_{\alpha,(F^{-1}\circ f)} = R_{f,T}$. This scheme gives the following corollary to Theorem 2.1.

Corollary 3.10 *Say that there is a proper class of Woodin cardinals and let κ strongly inaccessible be such that every ${}^\infty$-homogeneously Souslin set of reals is ${}^\infty$-homogeneously Souslin in V_κ. Then there is a forcing of size κ in whose extension the Continuum Hypothesis and Souslin Bounding hold.*

4 The Proof

The end-extension condition on the sets A^α_β in Theorem 2.1 gives that each of the forcings $\underset{\sim}{Q}_\alpha$ is semi-proper over the P_α-extension. Applying the preservation theorems in [3] we have that the forcing in Theorem 2.1 is well behaved.

Claim 4.1 *For $\alpha < \beta \leq \lambda$, P_β/P_α is semi-proper and κ-c.c., so $\aleph_1^{V^{\bar{P}}} = \aleph_1$, $\aleph_2^{V^{\bar{P}}} = \kappa$, $(2^{\aleph_0})^{V^{\bar{P}}} \leq \kappa \leq \aleph_2^{V^{\bar{P}}}$.*

It is shown in [2] that if κ is strongly inaccessible then an RCS limit of length κ of forcings of cardinality less than κ is κ-c.c.. The rest of Claim 4.1 follows from [3], Chapter X. The remaining point is to see that forcing with $\bar{P}$ does not add ω-sequences.

4.1 Semi-generics for sequences

The following are generalizations of ideas from Chapters V, X and XII of [3].

Definition 4.2 1. *We say $\bar{N} \in SEQ_\alpha(\chi)$ if*

   (a) $\alpha < \omega_1$,

   (b) $\bar{N} = \langle N_\beta : \beta < \alpha \rangle$,

   (c) each $N_\beta \prec (H(\chi), \in, <_\chi)$ is countable,

   (d) $\beta < \gamma \Rightarrow N_\beta \prec N_\gamma$,

   (e) $\beta < \alpha \Rightarrow \bar{N}\restriction(\beta+1) \in N_{\beta+1}$.

2. *Assume $P \in N_0$, $\bar{N} \in SEQ_\alpha(\chi)$. We say that $p \in P$ is $(\bar{N}, P)$-semi-generic if*

$$p \Vdash N_\delta[\underset{\sim}{G}_P] \cap \omega_1 = N_\delta \cap \omega_1$$

   for all $\delta < length(\bar{N})$. We say that $\bar{N}\restriction[\gamma, \beta]$ is P-generic if this holds for $\delta \in [\gamma, \beta]$.

3. *We say that P is α-semi-proper if for every $\bar{N} \in SEQ_\alpha(\chi)$ such that $P \in N_0$ and $p \in P \cap N_0$ there is a $(\bar{N}, P)$-semi generic $q \in P$ such that $p \geq q$.*

The proof of the theorem below is a straightforward combination of the proofs in [3] that CS iterations preserve α-properness and RCS iterations preserve semi-properness. Note that in the second part there is no need to find a $p \in \underset{\sim}{G}_P$ such that $(\bar{N}, P)$-semi-generic.

Theorem 4.3 *Let α be a countable ordinal.*

1. *If $\bar{P} = \langle P_\beta, \underset{\sim}{Q}_\beta : \beta < \gamma \rangle$ is an RCS limit of α-semi-proper forcings such that for each $\beta < \gamma$ there exists $n \in \omega$ such that*

$$1_{P_{\beta+n}} \Vdash |P_\beta| \leq \aleph_1,$$

then $\bar{P}$ is α-semi-proper.

2. *If $\bar{N} \in SEQ_\alpha(\chi)$, $\bar{P} \in N_0$, then*

$$1 \Vdash_P \bar{N}[\underset{\sim}{G}_P] = \langle N_\beta[\underset{\sim}{G}_P] : \beta \leq \alpha \rangle \in SEQ_\alpha(\chi)^{V[\underset{\sim}{G}_P]}.$$

To show that each $Q_{\beta,f}$ is α-semi-proper for all $\alpha < \omega_1$, we show that we can extend sequences of models in a suitable way.

Lemma 4.4 1. *Assume*

(a) *$\chi, \lambda_\alpha, \lambda_{\alpha+1}$ and $\langle A^\alpha_\beta : \beta < \omega_1 \rangle$ are as in the statment of Theorem 2.1,*

(b) *$\bar{N} \in SEQ_\alpha(\chi)$ and $\lambda_{\alpha+1} \in N_0$,*

(c) *each member of $\bar{N}$ is in C,*

(d) *$f : \omega_1 \to \omega_1$.*

Then we can find $\bar{N}' \in SEQ_\alpha(\chi)$ such that

(e) *each $N \subset N'$,*

(f) *each $N'_\eta \cap V_{\lambda_\alpha+1} = N_\eta \cap V_{\lambda_\alpha+1}$,*

(g) *each $N'_\eta \cap \lambda_{\alpha+1} \in A^\alpha_{f(N'_\eta \cap \omega_1)}$,*

(h) *$\bigcup\{N'_\eta : \eta \leq \alpha\} \cap \lambda_{\alpha+1} \in A^\alpha_{f(N' \cap \omega_1)}$.*

2. *If in addition, $f : \omega_1 \to \omega_1$, then each $Q_{\beta,f}$ is α-semi-proper for all $\alpha < \omega_1$.*

Proof: We prove the first part first. Fix $\bar{N}, \lambda_\alpha, \lambda_{\alpha+1}, \gamma$. If $\bar{N}$ has a last model, then we are done by extension assumption on the models in C from the statement of Theorem 2.1. For the limit case, let $N = \bigcup\{N_\eta : \eta < \alpha\}$, and apply the induction hypothesis to assume that we have already expanded to meet every condition except part (h). Let $N' \prec (H(\chi), \in, <_\chi)$ be such that

1. $N \subset N'$,
2. $N \cap V_{\lambda_\alpha+1} = N' \cap V_{\lambda_\alpha+1}$,
3. $N' \cap \lambda_{\alpha+1} \in A^\alpha_{f(N' \cap \omega_1)}$,

and let $X = (N' \cap \lambda_{\alpha+1}) \setminus N$. Let $h : \omega \to X$ be a bijection. Fix an increasing sequence $\langle \xi_i : i < \omega \rangle$ cofinal in α. Now let each $N'_\eta = Sk_{H(\chi),\in,<_\chi}(N_\eta \cup h[i_\eta])$, where i_η is the least i such that $\xi_i \geq \eta$. Since these are all finite extensions, each initial sequence of the new sequence is an element of the later models, and so $\bar{N}'$ is as desired.

The second part is easily proved by induction, using the first part. ■

Corollary 4.5 *For $\beta < \gamma \leq \kappa$, P_γ/P_β is α-semi-proper for every $\alpha < \omega_1$.*

4.2 Systems of models and conditions

Since $\bar{P}$ forces κ to be ω_2, we need to show only that each initial segment P_β does not add ω-sequences. To do this, we choose a pair $(M, \bar{N})$ with the properties below, where M is a countable elementary submodel of $H(\chi)$, and find a condition in P_β which extends an M-generic filter.

Definition 4.6 *We say that $(M, \bar{N}, p, q)$ is a $(\bar{Q}, \alpha, \beta)$-system if (letting $\delta = o.t.(M \cap \alpha)$ and $\gamma = o.t.(M \cap \beta)$):*

1. *$M \prec (H(\chi), \in, <_\chi)$ is in C,*
2. *$\bar{Q}$, α, β belong to M,*
3. *$\alpha \leq \beta$,*
4. *$\bar{N} \in SEQ_\gamma$,*
5. *for each $\alpha' \in M \cap (\alpha + 1)$, $q{\restriction}\alpha' \in P_{\alpha'}$ is $(\bar{N}{\restriction}[\delta', \gamma), P_{\alpha'})$-semi-generic, where $\delta' = o.t.(M \cap \alpha')$,*
6. *q forces a value to $\underset{\sim}{G}_{P_\alpha} \cap M$,*
7. *if $\eta \in [\delta, \gamma) \cap M$ then*
   - *(a) for all $\beta \in N_{o.t(\eta \cap M)} \cap \omega_1$, $M \cap \lambda_{\eta+1} \in A^\eta_\beta$,*
   - *(b) $M \cap H(\lambda_{\eta+1}) \in N_{o.t.(\eta \cap M)+1}$,*
8. *$p \in P_\beta$, $p{\restriction}\alpha \geq q$ and for all $\eta \in [\delta, \gamma) \cap M$, $p{\restriction}\delta \in N_{o.t.(\delta \cap M)}$.*

Definition 4.7 *Let $M \prec (H(\chi), \in, <_\chi)$ and $\beta < \kappa$ with $\beta, P \in M$. A condition $p \in M$ is a potential-M-generic for P_β if there exists a $g \subset P \cap M$ be M-generic for P_β such that for all $\alpha \in M \cap \beta$,*

$$1 \Vdash (M \cap \lambda_{\alpha+1} \in A^\alpha_{\underset{\sim}{f}_\alpha(M \cap \omega_1)}) \Rightarrow p(\alpha) = g(\alpha).$$

Given a $g \subset P \cap M$ be M-generic for P_β, one can easily build a corresponding potential-M-generic p, letting $p(\alpha)$ be the empty condition when

$$M \cap \lambda_{\alpha+1} \notin A^\alpha_{\underset{\sim}{f}_\alpha(M \cap \omega_1)}.$$

Lemma 4.8 *If for every $\beta < \kappa$, $p_0 \in P_\beta$ and $x \in H(\chi)$ there is a $(\bar{Q}, \beta, \beta)$-system $(M, \bar{N}, p, q)$ with $p_0, x \in M$ and $p \leq p_0$ a potential-M-generic, then the forcing $\bar{P}$ does not add ω-sequences.*

Proof: The condition q forces that the realization of p will in fact be M-generic, and so every name in M for an ω-sequence will take a value in V. ■

Given a countable $X \prec H(\chi)$ with $\lambda \in X$, and given $\eta < \lambda$, we say that $Y \prec H(\chi)$ is a minimal (η, λ)-extension of X if the following hold.

1. $X \cap \eta = Y \cap \eta$.
2. $Y = Sk_{(H(\chi), \in, <_\chi)}(X \cup A)$, for some $A \subset \lambda$.

The fact about these extensions that we will use is given in the following lemma.

Lemma 4.9 *Let $X \prec H(\chi)$, and let γ be a regular cardinal in $X \setminus \lambda^+$. Let Y be a minimal (η, λ)-extension of X for any $\eta < \gamma$. Then $X \cap \gamma$ is cofinal in $Y \cap \gamma$.*

Proof: Since γ is regular, each $f : [\lambda]^{<\omega_1} \to \gamma$ has bounded range below γ. If f in X then this bound exists in X. ■

Lemma 4.10 *For any $x \in H(\chi)$,$\alpha \leq \beta < \kappa$ and $p \in P_\alpha$, there is a $(\bar{Q}, 0, \beta)$-system $(M, \bar{N}, p, \emptyset)$ with $p, x \in M$.*

Proof: Say that $(M, \bar{N})$ is a (γ, p)-approximation if

1. $M \prec (H(\chi), \in, <_\chi)$ is in C
2. $\{\alpha, \beta, p, x\} \in M$,
3. $\gamma \in M \cap (\beta + 1)$,
4. $\bar{N} = \langle N_\xi : \xi \in M \cap \gamma \rangle$,
5. $N_\xi \prec (H(\chi), \in, <_\chi)$ is countable,
6. $\langle N_\xi : \xi \in M \cap (\eta + 1) \rangle \in N_{\eta+1}$ for $\eta \in M \cap \gamma$,
7. if $\eta \in [\delta, \gamma) \cap M$ then
   (a) for all $\beta \in N_{o.t(\eta \cap M)} \cap \omega_1$, $M \cap \lambda_{\eta+1} \in A^\eta_\beta$,
   (b) $M \cap H(\lambda_{\eta+1}) \in N_{o.t.(\eta \cap M)+1}$,

Let K be the set of $(\gamma, M, \bar{N})$ such that $(M, \bar{N})$ is a (γ, p)-approximation.
Let $(\gamma_1, M_1, \bar{N}_1) <_K (\gamma_2, M_2, \bar{N}_2)$ iff

1. $\gamma_1 < \gamma_2$,
2. $M_1 \prec M_2$,

3. $M_1 \cap H(\lambda_{\gamma_1}) = M_2 \cap H(\lambda_{\gamma_1})$,

4. $\bar{N}_1$ is an initial segment of $\bar{N}_2$.

Claim 1: $K \neq \emptyset$. Let $\{x, p\} \in M \prec (H(\chi), \in, <_\chi)$, M countable, $\gamma = 0$.
Claim 2: $<_K$ is a partial order. Easy
Claim 3: If $\langle(\gamma_n, M_n, \bar{N}_n) : n < \omega\rangle$ is $<_K$-increasing it has a limit upper bound $\gamma = min\{\bigcup_{n<\omega} M_n \cap (\beta+1) \setminus (\bigcup_{n<\omega} \gamma_n)\}$, $M = \bigcup_{n<\omega} M_n$, $\bar{N}$ is the limit of $\bar{N}_n$ (i.e., the minimal common extension).
Claim 4: If $(\gamma, M, \bar{N}) \in K$ then we can find $M', \bar{N}'$ such that $(\gamma, M, \bar{N}) <_K (\gamma+1, M', \bar{N}')$. This follows from the extendibility assumption in Theorem 2.1.

Try to build a K-increasing ω_1-sequence $\langle(\gamma_\xi, M_\xi, \bar{N}_\xi) : \xi < \omega_1\rangle$ by induction, using Claim 1 for $\xi = 0$, taking minimal $(2^{\lambda_\gamma}, \lambda_{\gamma+1})$-extensions as in Claim 4 for successor stages and limits as in Claim 3 at limit stages. If we cannot continue to define our sequence for some $\xi < \omega_1$, by Claims 1,3 and 4 the only reason can be that ξ is a successor and $\gamma_{\xi-1} = \beta$. Then we are done, letting $N'_{o.t.(\alpha\cap M)} = N_\alpha$ for $\alpha \in M \cap (\beta+1)$.

The other case is that we succeed in building such a sequence. In this case, let $M = \bigcup_{\xi<\omega_1} M_\xi$. Let

$$\gamma^* = min(M \setminus \bigcup_{\xi<\omega_1} \gamma_\xi).$$

We will show that the cofinality of $M \cap \gamma^*$ is countable, which is a contradiction since $\langle\gamma_\xi : \xi < \omega_1\rangle$ is increasing and cofinal in it.

First, if $\gamma^* \leq \lambda_{\gamma_\xi}$ for some γ_ξ with $\gamma^* \in M_\xi$, then $M \cap \gamma^* = M_\xi \cap \gamma^*$, which is countable. Otherwise, $\gamma^* = \lambda_{\gamma^*}$. If γ^* is singular, let α^* be the cofinality of γ^*. Then $\alpha^* < \lambda_{\gamma_\xi}$ for some γ_ξ with $\{\alpha^*, \gamma^*\} \in M_\xi$. There exists a cofinal map $f : \alpha^* \to \gamma^*$ in M_ξ. Since $\alpha^* \cap M = \alpha^* \cap M_\xi$, $f[M_\xi \cap \alpha^*]$ is a countable set cofinal in $M \cap \gamma^*$. The last remaining case is that γ^* is a regular limit cardinal. Let ξ be least with $\gamma^* \in M_\xi$, and fix a cofinal sequence in $M_\xi \cap \gamma^*$. By Lemma 4.9 and induction, this sequence is cofinal in $M \cap \gamma^*$. ■

We finish by applying the following lemma to the case $\alpha = 0, \gamma = \beta$, from Lemma 4.10. By Lemma 4.8, then, we are done.

Lemma 4.11 *Assume $(M, \bar{N}, p, q)$ is a $(\bar{Q}, \alpha, \beta)$-system and $\alpha \leq \gamma \leq \beta$. Then for some q' we have*

1. $q' \in P_\gamma$,

2. $q' \restriction \alpha = q$,

3. $(M, \bar{N}, p, q')$ is a $(\bar{Q}, \gamma, \beta)$-system.

Proof: Induct primarily on $o.t.((\beta \setminus \alpha) \cap M)$, and secondarily on $\gamma \in [\alpha, \beta] \cap M$. When $\alpha = \beta$ there is nothing to show, and increasing γ by one follows from the fact that $\bar{P}$ is η-semi-proper for all countable η, as does increasing $o.t.((\beta \setminus \alpha) \cap M)$ arbitrarily while fixing γ.

For the case where γ is a limit, fix an increasing cofinal sequence $\langle \delta_i : i < \omega \rangle$ in $\gamma \cap M$. The point is that if $\gamma < \beta$ then we need to make sure that the resulting q' is $(\bar{N}\restriction[o.t.(M \cap \gamma), o.t.(M \cap \beta))$-semi-generic. So in this case, let $\langle \tau_i : i < \omega \rangle$ be a wellordering of the P_γ-names for countable ordinals in $\cup(\bar{N}\restriction[o.t.(M \cap \gamma), o.t.(M \cap \beta))$. Apply the induction hypothesis for each i as follows. Let $q_0 = q$, $p_0 = p$. Given q_i, let q_{i+1}, p_{i+1} satisfy

1. $q_{i+1} \in P_{\delta_{i+1}}$,
2. $q_{i+1}\restriction\delta_i = q_i$,
3. $p_{i+1} < p_i$,
4. $p_{i+1}\restriction\delta_{i+1} \geq q_{i+1}$,
5. $q_i \wedge p_{i+1} \Vdash \tau_i \in \omega_1 \cap \check{M}$ (if $\gamma < \beta$),
6. $(M, \bar{N}, p_{i+1}, q_{i+1})$ is a $(\bar{Q}, \delta_{i+1}, \beta)$-system.

Then the limit of the q_i's is the desired q'. That we can meet Condition 5 is a standard part of the proof that Revised Countable Support preserves semi-properness. In brief, let ξ be least such that $\tau \in N_\xi$. $p_{i+1}\restriction[\delta_i, \delta_{i+1})$ can be chosen to be a P_{δ_i}-name in N_ξ for a condition in $P_{\delta_{i+1}}/P_{\delta_i}$ such that each member of the P_{δ_i}-part of the antichain deciding τ forces that the realization of $p_{i+1}\restriction[\delta_i, \delta_{i+1})$ will be a corresponding $P_{\delta_{i+1}}/P_{\delta_i}$-part. Then since q_i is (N_ξ, P_{δ_i})-semi-generic, $q_i \wedge p_{i+1}$ will be as desired.

The case where $o.t.((\beta \setminus \alpha) \cap M)$ and γ are both limits is similar. Fix cofinal increasing sequences $\langle \delta_i : i < \omega \rangle$ and $\langle \eta_i : i < \omega \rangle$ in $\gamma \cap M$ and $(\beta \setminus \alpha) \cap M$ respectively, with $\delta_0 = \alpha$ and each $\delta_i \in N_{o.t.(M \cap \eta_i)}$. If $\gamma < \beta$, let $\eta_0 = \gamma$ and let $\langle \tau_i : i < \omega \rangle$ be a wellordering of the P_γ-names for countable ordinals in $\cup(\bar{N}\restriction[o.t.(M \cap \gamma), o.t.(M \cap \beta))$, such that $\tau_i \in N_{o.t.(M \cap \eta_i)}$. Apply the induction hypothesis for each i as follows. Let $q_0 = q$ and let $p_0 = p$. Given q_i, let q_{i+1}, p_{i+1} satisfy

1. $q_{i+1} \in P_{\delta_{i+1}}$,
2. $q_{i+1}\restriction\delta_i = q_i$,
3. $p_{i+1} < p_i$,
4. $p_{i+1}\restriction\delta_{i+1} \geq q_{i+1}$,
5. $q_i \wedge p_{i+1} \Vdash \tau_i \in \omega_1 \cap \check{M}$ (if $\gamma < \beta$),
6. $(M, \bar{N}, p_i, p_{i+1}\restriction\gamma)$ is a $(\bar{Q}, \gamma, \eta_{i+1})$-system (if $\gamma < \beta$),
7. $(M, \bar{N}, p_{i+1}, q_{i+1})$ is a $(\bar{Q}, \delta_{i+1}, \eta_{i+1})$-system.

To meet condition 6, the induction hypothesis tells us that such a $p_{i+1} \restriction \gamma$ exists, and then the fact that $N_{o.t.(M \cap \eta_{i+1})} \prec H(\chi)$ gives that such a condition exists in $N_{o.t.(M \cap \eta_{i+1})}$, meeting condition 8 of Definition 4.6. Continuing in this way for ω stages completes the construction, and the limit of the q_i's is the desired q'. ■

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5100 Randall Lane
Bethesda, MD 20816
USA

larson@ml.kva.se

Institute of Mathematics
Hebrew University
91904 Jerusalem
Israel

Department of Mathematics
Rutgers University
New Brunswick, NJ 08903
USA

shelah@math.huji.ac.il