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REPORT No. 5, 2000/2001

ISSN 1103-467X

ISRN IML-R- -5-00/01- -SE



INSTITUT MITTAG-LEFFLER
THE ROYAL SWEDISH ACADEMY OF SCIENCES

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November 27, 2000

Abstract

In 1948 Miroslav Katětov showed that if the cube X^3 of a compact space X satisfies the separation axiom T_5 then X must be metrizable. He asked whether X^3 can be replaced by X^2 in this metrization result. In this note we prove the consistency of this implication.

MSC 2000: Primary 54E35; Secondary 03E35, 03E65, 54E45

Keywords: compact, metrizable, T_5 , forcing

1 Introduction

In his 1948 paper [9], Miroslav Katětov proved that if the cube of a compact space is T_5 (also called completely normal or hereditarily normal)¹ then the space must be metrizable. He asked if the cube can be replaced by the square in this statement. It has been known for some time that MA_{ω_1} and CH each give counterexamples to the implication, and more recently, a considerable weakening of CH was proved to be sufficient for a counterexample (see [19, 4, 18]). In this note we show the consistency of Katětov's metrization statement: $X^2 T_5 \Rightarrow X$ metrizable. The key feature of our proof is the combinatorial analysis of particular forcing axioms corresponding to the maximal amount of the forcing axioms MA_{ω_1} , PFA, MM and the \mathbb{P}_{max} axiom (*) compatible with the existence of a particular type of Souslin tree (see [1, 15, 16, 12, 13, 22, 27]). It turns out that for solving Katětov's problem it suffices to analyze only the maximal amount of MA_{ω_1} compatible with the existence of this type of tree. Since MA_{ω_1} is by far the most familiar forcing axiom (see [2]), in order to make this paper accessible to a wider audience we have decided to extract only this part of our analysis from the more general picture that will be dealt with in subsequent papers.

It has been known for some time that the metrizability of a compactum X whose square is T_5 is closely tied with two seemingly contradictory statements,

*This work was done while the authors were in residence at the Mittag-Leffler Institute. We thank the Institute for its hospitality.

¹Recall that the separation axiom T_5 says that every two separated sets A and B (i.e., $\bar{A} \cap B = \bar{B} \cap A = \emptyset$) can be separated by disjoint open sets.

the assumption that there are more sets of countable ordinals than reals (which has been used quite successfully in analyzing the class of separable first countable normal spaces ever since Burton Jones's work [8] in 1937), and the assumption that a sufficient fragment of MA_{ω_1} holds, in particular the fragment that would ensure the separability of perfectly normal compacta (see [11, 4, 26]). Finding the right form of Jones's hypothesis and the right form of MA_{ω_1} , consistent with each other yet strong enough to give enough information about T_5 spaces to solve Katětov's problem, is the difficulty which has kept the problem open for so long. It is reasonable to expect that a continuation of this work will eventually result in a much better understanding of the whole class of compact T_5 spaces. For example, the knowledge already accumulated suggests that separable - or even c.c.c. - spaces from this class must be closely tied to the unit interval, almost as closely tied, for instance, as the split interval $[0, 1] \times \{0, 1\}$. We also hope that this work will motivate further study of fragments of forcings axioms surviving mild forcing extensions. In our case, the extension is by a Souslin tree. Equally attractive are forcing extensions by measure algebras, as these are also quite relevant to the structure of compact T_5 spaces, as shown for example in [26], §4. We refer the reader to the works [20, 14, 25, 17, 6] for more on the preservation of forcing axioms under measure algebra extensions.

2 Notation

We let trees grow downwards, so that the tree ordering corresponds with the forcing ordering. If S a tree and $s \in S$, then $\text{lev}_S(s) = \alpha$ means that s is on the α th level of S , i.e., that the order type of $\{x \in S \mid x > s\}$ is α reversed. So, for example, if $s \leq s'$ in S , then $\text{lev}_S(s) \geq \text{lev}_S(s')$. The meet of a finite subset $a \subset S$ is the least node which is above each member of a . For members p, p' of a partial order \mathbb{P} , $p \perp p'$ means that p and p' are incompatible, and if they are compatible, $p \wedge p'$ denotes the smaller of the two, i.e., p if $p \leq p'$. The set of unordered n -tuples (finite sets) from a set X is denoted $[X]^n$ ($[X]^{<\omega}$). Given sets A and B , $A \otimes B$ is the set of unordered pairs from $A \cup B$ with one member in A and the other in B . For a function f and a subset a of the domain of f , $f|a$ is the function with domain a which agrees with f on all points in a . If A is a subset of a topological space X , \bar{A} is the closure of A in X .

We use the usual forcing terminology (see, e.g. [10]). Every object in an extension by a forcing \mathbb{P} is represented by a \mathbb{P} -name in the ground model. The class of \mathbb{P} -names is defined recursively, each name being a set of pairs (p, τ) where p is a condition in \mathbb{P} and τ is a \mathbb{P} -name which has already been defined. The pair (p, τ) being a member of a name σ means that the condition p forces that the realization of τ will be a member of the realization of σ . For each set x in the ground model, \check{x} is the canonical name for x , i.e., the set of pairs $(1_{\mathbb{P}}, \check{a})$ for $a \in x$, where $1_{\mathbb{P}}$ is the empty condition in \mathbb{P} . The names that we refer to in this paper are all names for subsets of sets from the ground model, and as such are composed of pairs of the form (p, \check{x}) .

3 Souslin's axiom

A *coherent* tree is a downward closed subtree S of $\omega^{<\omega_1}$ with the property that $\{\xi \in \text{dom}(s) \cap \text{dom}(t) : s(\xi) \neq t(\xi)\}$ is finite for all $s, t \in S$. In this paper we work with coherent Souslin trees, which are Souslin trees given by a coherent family of function in $\omega^{<\omega_1}$ closed under finite modifications. For S a coherent Souslin tree and s, t on the same (η th) level of S , there is a canonical isomorphism π_{st}^S between the cones below s and t , defined by letting $\pi_{st}^S(s')(\alpha)$ be $t(\alpha)$ if $\alpha < \eta$ and $s'(\alpha)$ otherwise, for each $s' \leq s$.

We let SA_{ω_1} (where SA stands for ‘‘Souslin’s Axiom’’) denote the statement that there is a coherent Souslin tree S such that for all posets \mathbb{P} with $\mathbb{P} \times S$ c.c.c., and any collection D_ξ ($\xi < \omega_1$) of dense open subsets of \mathbb{P} , there is a filter $G \subset \mathbb{P}$ such that $G \cap D_\xi \neq \emptyset$ for all $\xi < \omega_1$. In our discussion of SA_{ω_1} we let S refer to such a coherent tree.

To obtain a model of SA_{ω_1} , start with a coherent Souslin tree S and a cardinal θ such that $\theta^{\omega_1} = \theta$, and build a finite support iteration of length θ of c.c.c. posets of size \aleph_1 . This is worked out in [1, 12]. Stronger versions of the axiom appear in [15, 16, 12, 13, 22].

In the following section we shall prove that under SA_{ω_1} a considerable amount of MA_{ω_1} is true after forcing with S . It turns out that the amount of MA_{ω_1} forced by S over SA_{ω_1} is sufficient to give the metrizability of any compact space whose square is T_5 .

4 Rectangularly refinable partitions

Recall the notion of a c.c.c. partition $[\omega_1]^2 = K_0 \cup K_1$ from [24]: the partition is said to be c.c.c. if for all sequences $a_\xi \in [\omega_1]^{<\omega}$ ($\xi < \omega_1$) either some $[a_\xi]^2 \not\subset K_0$ or there is a pair $\xi_0 \neq \xi_1$ such that $[a_{\xi_0} \cup a_{\xi_1}]^2 \subset K_0$. Let \mathcal{K}_2 be the statement that every c.c.c. partition on pairs from ω_1 has an uncountable homogeneous subset. The statement \mathcal{K}_2 is clearly a consequence of MA_{ω_1} , and it is well known that \mathcal{K}_2 implies many of the consequences of MA_{ω_1} . In fact, it is still unknown whether \mathcal{K}_2 implies MA_{ω_1} , and it is this question which has motivated us to investigate this subject (see [24], §7; [13]). In this section we shall show that forcing with the coherent Souslin tree S over a model of SA_{ω_1} gives \mathcal{K}_2 for a class of c.c.c. partitions on $[\omega_1]^2$ which is especially relevant to Katětov’s problem.

4.1 Definition. A partition $P \subset [\omega_1]^2$ satisfies the *rectangle refining property* if for all uncountable $A, B \subset \omega_1$ there are uncountable $A' \subset A$, $B' \subset B$ such that $A \otimes B \subset P$.

We let $\mathcal{K}_2(\text{rec})$ denote the statement that every c.c.c. partition on $[\omega_1]^2$ with the rectangle refining property has an uncountable homogeneous set.

Say that S is a Souslin tree and τ is an S -name for a partition on $[\omega_1]^2$. Since S is c.c.c. and adds no reals, for each countable ordinal α , $\tau \cap [\alpha]^2$ will be a set in the ground model. We use this fact to define a convenient form for τ , letting S_τ be the set of pairs $(t, \alpha) \in T \times \omega_1$ such that for some x in the ground

model t forces $\tau \cap [\check{\alpha} + 1]^2 = \check{x}$. This allows us to think of τ as a partition on pairs from the set S_τ in the ground model. We let \mathcal{P}_τ denote the partition on $[S_\tau]^{<\omega}$ defined by letting $a \in \mathcal{P}_\tau$ if and only if for every pair $\{(t, \alpha), (t', \alpha')\} \subset a$, either $t \perp t'$ or $t \wedge t'$ forces $\{\check{\alpha}, \check{\alpha}'\} \in \tau$. Then an uncountable 0-homogeneous set for \mathcal{P}_τ , assuming that S is still Souslin, gives an S -name for an uncountable 0-homogeneous subset of τ .

4.2 Definition. Let S be a Souslin tree and τ a S -name for a partition on $[\omega_1]^2$. In a forcing extension by S , an \mathcal{P}_τ -stalk is a sequence of finite sets $\langle a_\alpha \in \mathcal{P}_\tau : \alpha < \omega_1 \rangle$ such that each a_α has the same size n , and, letting each $a_\alpha = \langle (q_\alpha^i, \beta_\alpha^i) : i < n \rangle$,

1. there is a decreasing path p_α ($\alpha < \omega_1$) through S such that each $q_\alpha^i \leq p_\alpha$, and such that for all $\alpha < \omega_1, i < n$,

$$\text{lev}_S(q_\alpha^i) < \text{lev}_S(p_{\alpha+1}),$$

2. for each α , the relationships in S between the finite meets of the q_α^i 's is the same.

A \mathcal{P}_τ -stalk induces a partition $[\omega_1]^2 = K_0 \cup K_1$ by letting $\{\alpha_0, \alpha_1\} \in K_0$ if and only if a_{α_0} and a_{α_1} are comatible in \mathcal{P}_τ .

We use the following standard fact about coherent trees.

Lemma 4.3. *If S is a coherent tree with a cofinal branch, then every uncountable subset of S contains a set q_α ($\alpha < \omega_1$) such that there exists a strictly decreasing chain $\langle p_\alpha \mid \alpha < \omega_1 \rangle \subset S$ with the property that for all α , $q_\alpha \leq_S p_\alpha$.*

Proof: Since there is a branch through S , we can pick a function $f \in \omega^{\omega_1}$ such that the initial segments of f are all in S . Let A be an uncountable subset of S . Each member of A differs from f in finitely many places, so we can shrink A so that these sets form a Δ -system. Letting r be the root of the Δ -system, we can shrink A again so that each member of A takes the same values on r . Let f' be the finite modification of f created by letting f' take these same values on r , and letting f' and f agree everywhere else. Since for each $\alpha < \omega_1$ there is a $q \in A$ which extends $f' \upharpoonright \alpha$, we can pick q_α 's as desired, taking the corresponding p_α 's from the initial segments of f' . \square

Lemma 4.4. *([13]) Say that S is a coherent Souslin tree and that τ is a S -name for a partition on $[\omega_1]^2$. Then if forcing with \mathcal{P}_τ does not preserve the Souslinity of S , then after forcing with S there is a finite collection of \mathcal{P}_τ -stalks whose product is not c.c.c.*

Proof: First, if forcing with \mathcal{P}_τ does not preserve the Souslinity of S , then $\mathcal{P}_\tau \times S$ is not c.c.c., and so \mathcal{P}_τ is not c.c.c. after forcing with S . Fix a sequence a_α ($\alpha < \omega_1$) of incomatible n -tuples in \mathcal{P}_τ in the forcing extension by S , for some integer n .

Apply Lemma 4.3 to the i th members of our sequence for each $i < n$, thinning the antichain appropriately each time. For each α , let $a_\alpha = \langle (q_\alpha^i, \beta_\alpha^i) : i < n \rangle$. Since there are only finitely many such patterns, we may assume that for each α , the relationships in S between the finite meets of the q_α^i 's is the same. We may also fix a function $f : n \rightarrow m$ for some $m \leq n$, and descending sequences $\langle p_\alpha^j : \alpha < \omega_1 \rangle$ through S for each $j < m$ such that

1. for all $i < n$, $\alpha < \omega_1$,

$$q_\alpha^i \leq p_\alpha^{f(i)},$$

2. for all $i, j < n$, $\alpha < \beta < \omega_1$,

$$\text{lev}_S(q_\alpha^i) < \text{lev}_S(p_\beta^j).$$

Then for each $j < m$, $\langle \{(q_\alpha^i, \beta_\alpha^i) \mid f(i) = j\} : \alpha < \omega_1 \rangle$ is a \mathcal{P}_τ -stalk, and since the a_α 's are pairwise incompatible in \mathcal{P}_τ , the product of these stalks is not c.c.c. \square

We are now ready to state the key technical lemma of this paper.

Lemma 4.5. *If S is a coherent Souslin tree and τ is an S -name for a rectangle refining partition on pairs from ω_1 , then in any forcing extension by S , any \mathcal{P}_τ -stalk contains an uncountable 0-homogeneous set in its induced partition. Therefore, any finite product of \mathcal{P}_τ -stalks in this extension is c.c.c..*

Proof: Assume that we have a S -name σ for an \mathcal{P}_τ -stalk. Since S is coherent, the path corresponding to the \mathcal{P}_τ -stalk will be a translate (i.e., an image under some $\pi_{s_i}^S$) of the generic path, and we may assume that the path will be the same as the generic path.² This gives us an uncountable set \mathcal{A} of objects (p, q, a) , where $a = \langle (t_i, \beta_i) \in S_\tau : i < n \rangle$, such that

1. q forces that $\langle (t_i, \beta_i) : i < n \rangle \in \sigma$,
2. the meet of the t_i 's is below p .
3. $q \leq p$ and $\text{lev}_S(q) > \text{lev}_S(t_i)$ for all $i < n$.

Furthermore, we may assume that for each pair $(p, q, a), (p', q', a') \in \mathcal{A}$ with $a = \langle (t_i, \beta_i) \in S_\tau : i < n \rangle, a' = \langle (t'_i, \beta'_i) \in S_\tau : i < n \rangle$, either $\text{lev}_T(p) > \text{lev}_T(q')$ or $\text{lev}_T(p') > \text{lev}_S(q)$. We may also fix $m \leq n$ such that for all $(p, q, a) \in \mathcal{A}$ with

$$a = \langle (t_i, \beta_i) \in S_\tau : i < n \rangle,$$

$$\{t_i : i < m\} = \{t_i \geq q : i < n\}.$$

Let s_0 be such that the set of p such that there exists $(p, q, a) \in \mathcal{A}$ is dense below s_0 . We will refine \mathcal{A} in $m \times n$ stages. In the k th stage, corresponding

²This relies on the fact that S is coherent, and so in the extension by S , all paths through S will be images of one another under the isomorphisms $\pi_{s_i}^S$.

to $i_0 < m$, $i_1 < n$, we refine \mathcal{A} so that for all $(p, q, a), (p', q', a') \in \mathcal{A}$ with $a = \langle (t_i, \beta_i) \in S_\tau : i < n \rangle, a' = \langle (t_i, \beta_i) \in S_\tau : i < n \rangle$ and $p \leq q'$, t_{i_1} forces that β_{i_1} and β'_{i_0} are compatible in τ . We do this keeping \mathcal{A} uncountable, and then pick $s_{k+1} \leq s_k$ such that the set of p such that there exists $(p, q, a) \in \mathcal{A}$ is dense below s_{k+1} , and restrict \mathcal{A} to triples whose p is below s_{k+1} . After $m \times n$ stages we will have a name for an uncountable substalk of σ all of whose pairs are compatible, and so we will be done.

So fix i_0, i_1, s_k . \mathcal{A} induces two names ρ_0 and ρ_1 for uncountable subsets of τ , given respectively by the set of pairs (t_{i_1}, β_{i_1}) and the set of pairs (t_{i_0}, β_{i_0}) from the members of \mathcal{A} . Since τ has the rectangle refining property, there are names ρ'_0 and ρ'_1 for an uncountable compatible rectangle, with each ρ'_i forced to be a subset of the corresponding ρ_i . Shrink \mathcal{A} first so that for every member of \mathcal{A} there is a t' below its t_{i_0} forcing that its β_{i_0} is in ρ'_0 , and then do the same for i_1 . Since each $(t, \beta) \in S_\tau$ decides $\tau \cap [\beta + 1]^2$, \mathcal{A} now has the property that for every $a, a' \in \mathcal{A}$, if $p \leq q'$, then t_{i_1} forces that β_{i_1} and β'_{i_0} are compatible in τ , for $p, q', t_{i_1}, \beta_{i_1}$ and β'_{i_0} the corresponding members of a and a' . Lastly, since \mathcal{A} remains uncountable, the corresponding p 's are dense below some $s \leq s_k$, which we choose as s_{k+1} .

For the second part of the lemma, we use the standard argument that \mathcal{K}_2 implies that the product of c.c.c. partial orders is c.c.c. (see [2]). For any uncountable subset of a finite product of \mathcal{P}_τ -stalks after forcing with S , we can refine along each coordinate so that its members are pairwise compatible. Having done this for each coordinate, we have an uncountable 0-homogeneous set in the product, showing that the original subset was not an antichain \square

Putting these two lemmas together gives the following.

Lemma 4.6. *Say that S is a coherent Souslin tree, and that τ is an S -name for a rectangle refining partition on $[\omega_1]^2$. Then forcing with \mathcal{P}_τ preserves that S is Souslin.*

The following result summarizes the information gathered in this section, and will be used in the following sections.

Theorem 4.7 (SA $_{\omega_1}$). *Forcing with the Souslin tree S as in the statement of SA $_{\omega_1}$ produces a model of $\mathcal{K}_2(rec)$.*

4.8 Question. Is \mathcal{K}_2 true after forcing with the Souslin tree S in a model of SA $_{\omega_1}$?

5 Subspaces of first countable compact spaces

The purpose of this section is to prove a result which shows that $\mathcal{K}_2(rec)$ has enough strength to impose considerable structure on compact spaces with T_5 squares. It is this result that will move the difficulty of Katětov's problem to some questions about the real line which we consider in the next section.

Theorem 5.1 ($\mathcal{K}_2(\text{rec})$). *The following are equivalent for every space X which has a first countable compactification.*

- (1) *For every family \mathcal{U} of open subsets of X there is a countable subfamily $\mathcal{U}_0 \subset \mathcal{U}$ such that $\bigcup \mathcal{U}_0 = \bigcup \mathcal{U}$.*
- (2) *For every subset Y of X there is a countable $Y_0 \subset Y$ such that $\bar{Y}_0 = \bar{Y}$.*

Proof: To prove the implication from (1) to (2) let Y be a given uncountable subset of X . If there is no countable $Y_0 \subset Y$ such that $\bar{Y}_0 = \bar{Y}$, then by shrinking Y we may assume that Y comes with a wellordering $<_w$ of ordertype ω_1 such that each $y \in Y$ is not in $\overline{\{x \in Y \mid x <_w y\}}$. For each $y \in Y$ pick an open set U_y such that $y \in U_y$ and $\bar{U}_y \cap \{x \in Y \mid x <_w y\} \neq \emptyset$. Consider the following partition $[Y]^2 = K_0 \cup K_1$:

$$\{x, y\} \in K_0 \text{ iff } y \notin U_x.$$

Since we have $\mathcal{K}_2(\text{rec})$ at our disposal, the proof of the implication (1) \Rightarrow (2) is finished once we show that this partition satisfies the rectangle refining property. So let A and B be uncountable subsets of Y . By (1), there is a complete accumulation point z of B such that $z \in B$. Since z has a countable neighborhood base and since $z \notin \bar{U}_x$ for all $x \in Y$ with $x >_w z$, there is a neighborhood V of z such that

$$A_0 = \{x \in A : V \cap \bar{U}_x = \emptyset\}$$

is uncountable. Let $B_0 = B \cap V$. By the assumption that z is a complete accumulation point of B , B_0 is uncountable. Then $\{x, y\} \in K_0$ for all $x \in A$ and $y \in B$.

To prove the implication from (2) to (1), let \mathcal{U} be a given collection of open subsets of X such that $\bigcup \mathcal{U}_0 \neq \bigcup \mathcal{U}$ for all countable $\mathcal{U}_0 \subset \mathcal{U}$. Then we can fix a set $Y \subset X$, a wellordering $<_w$ of Y of ordertype ω_1 , and for each $x \in Y$ a member $U_x \in \mathcal{U}$ such that $x \in U_x$ but $y \notin U_x$ for all $y >_w x$. Let γX be a first countable compact space containing X . We can replace U_x with an open subset of γX whose trace on Y is the same as the trace of our original U_x , thus assuring that U_x is open in γX . For each $x \in Y$ choose a set V_x open in γX such that $x \in V_x \subset \bar{V}_x \subset U_x$, and consider $[Y]^2 = K_0 \cup K_1$ as follows:

$$\{x, y\} \in K_0 \text{ iff } x \notin V_y.$$

Again it remains only to check that this partition satisfies the rectangle refining property, so let A and B be uncountable subsets of Y . Let $z \in \gamma X$ be a condensation point of A . Note that $z \notin \bar{V}_y$ for any $y \in Y$, so since by our assumption z has a countable neighborhood base in γX we can find an open neighborhood V of z such that

$$B_0 = \{y \in B \mid V \cap \bar{V}_y = \emptyset\}$$

is uncountable. Let $A_0 = V \cap A$. By the assumption that z is a complete accumulation point of A , the set A_0 is uncountable, and

$$\{x, y\} \in K_0 \text{ for all } x \in A_0, y \in B_0.$$

This completes the proof. \square

5.2 Remark. Note that in the above proof all we need to assume is that the ambient space γX is first countable and Lindelöf.

6 Subspaces of the real line

Katětov [9] showed that if the square of a compact space X is T_5 then X is *perfect*, i.e., closed subsets of X are G_δ . Note that every such X is first countable and satisfies hypothesis (1) of Theorem 5.1. So under $\mathcal{K}_2(\text{rec})$ every compactum X whose square is T_5 satisfies (2). In particular, X is separable. We shall need the following classical facts.

Lemma 6.1. ([21], see also [3]) *A compact space is metrizable if and only its diagonal is a G_δ subset of its square.*

Proof: To prove the nontrivial direction, suppose we are given a compact space X^2 whose diagonal Δ can be written as the intersection $\bigcap_{n=0}^{\infty} G_n$ of open subsets of X^2 . Since X^2 is normal we may assume that in fact $\Delta = \bigcap_{n=0}^{\infty} \bar{G}_n$. For each n choose $f_n: X^2 \rightarrow [0, 1]$ such that $f_n[\Delta] = \{0\}$ and $f_n[X^2 \setminus G_n] \subset \{1\}$. Finally, define $d: X^2 \rightarrow [0, 1]$ by

$$d(x, y) = \sum_{n=0}^{\infty} \frac{\max\{|f_n(x, z) - f_n(y, z)| : z \in X\}}{2^n}.$$

It is easily checked that d is a metric on X which generates the original topology on X . \square

Lemma 6.2. ([8, 5]) *If a separable, first countable normal space has an uncountable closed discrete subspace then there is an uncountable set of reals A such that every subset of A is relatively G_δ .*

Proof: Let Z be a normal first countable space that can be written as $C \cup D$ where C is a countable dense subset of Z , D is a closed discrete subset of Z and $C \cap D = \emptyset$. For each $x \in D$, choose a subset a_x of C which converges to x . It suffices to show that the almost disjoint family $A = \{a_x : x \in D\}$ has the property that for every $B \subset A$ there is a $b \subset C$ such that for every $x \in D$, $x \in B$ if and only if $a_x \cap b$ is infinite. Given such a $B \subset A$, since Z is a normal space there is a continuous $f: Z \rightarrow [0, 1]$ such that $f(x) = 0$ for all $x \in D$ with $a_x \in B$ and $f(x) = 1$ for all $x \in D$ with $a_x \notin B$. Let $b = \{x \in C \mid f(x) < 1/2\}$. \square

Let X be a given compact first countable space and assume that $\mathcal{K}_2(\text{rec})$ holds. Note that if the diagonal of X is not G_δ then X^2 fails to satisfy the property (1) of Theorem 5.1. To see this, consider the family \mathcal{U} of all sets of the form $X^2 \setminus F$, where F is a closed G_δ superset of the diagonal. Applying $\mathcal{K}_2(\text{rec})$ to X^2 we conclude that there is a $Y \subset X^2$ such that $\bar{Y}_0 \neq \bar{Y}$ for all countable $Y_0 \subset Y$. Shrinking Y we may assume that it comes with a wellordering $<_w$ of ordertype ω_1 such that $y \notin \overline{\{x \in Y \mid x <_w y\}}$ for all $y \in Y$. Applying $\mathcal{K}_2(\text{rec})$ to Y via Theorem 5.1 we conclude that Y fails to satisfy (1). Let \mathcal{U} be a family of open subsets of Y such that $\bigcup \mathcal{U}_0 \neq \bigcup \mathcal{U}$ for all countable $\mathcal{U}_0 \subset \mathcal{U}$. Then we can find uncountable $Z \subset Y$ such that for each $x \in Z$ there is an element $U_x \in \mathcal{U}$ such that $y \notin U_x$ whenever $x <_w y$. It follows that Z is a discrete subspace of X^2 . So if X^2 were T_5 the space would be perfect by Katětov's result [9], and so it would satisfy the hypothesis (1), and therefore the hypothesis (2), of Theorem 5.1, as $\mathcal{K}_2(\text{rec})$ holds. In particular, X , and therefore X^2 , would be separable. Choosing a countable dense set C in X^2 disjoint from \bar{Z} and applying Lemma 6.2 to the subspace $C \cup Z$, we obtain an uncountable set of reals A satisfying the conclusion of the lemma. As customary, let $\mathfrak{q} = \aleph_1^3$ denote the statement that such a set of reals cannot be found, i.e., the statement that for every uncountable $A \subset \mathbb{R}$ there is a $B \subset A$ such that $B \neq G \cap A$ for any G_δ -set $G \subset \mathbb{R}$. So we have established the following fact.

Theorem 6.3 ($\mathcal{K}_2(\text{rec})$ & $\mathfrak{q} = \aleph_1$). *A compact space is metrizable if and only if its square is T_5 .*

6.4 Remark. The proof of Lemma 6.2 shows that if there is a separable, first countable normal space with an uncountable closed discrete subspace, then there is a family y_α ($\alpha < \omega_1$) of pairwise almost disjoint subset of ω such that for every set $A \subset \omega_1$ there is a $z \subset \omega$ such that

$$A = \{\alpha < \omega_1 \mid z \cap y_\alpha \text{ is infinite}\}.$$

The existence of such a family is an equivalent form of $\mathfrak{q} \neq \aleph_1$. This is the form used most often in set theory [7], and it is commonly used in topology as well (see, e.g. [23]). For example, it is essentially shown in [4] that $\mathfrak{q} \neq \aleph_1$ implies that there is an uncountable $E \subset [0, 1]$ such that the square of $S(E)$ is T_5 .⁴ Thus $\mathfrak{q} \neq \aleph_1$ implies the negative answer to Katětov's question.

Now the consistency of the positive answer to Katětov's question follows from the following fact.

Lemma 6.5. (*[13]*) $\mathfrak{q} = \aleph_1$ holds after forcing with a Souslin tree.

Proof: Let S be a Souslin tree, and let τ_α ($\alpha < \omega_1$) be S -names for an ω_1 -sequence of pairwise almost disjoint subsets of ω . Since S is c.c.c. and has height ω_1 , and since forcing with S doesn't add reals, we may assume that there

³ $\mathfrak{q} = \min\{\theta \mid \forall A \in [\mathbb{R}]^\theta \exists B \subset A \forall G_\delta\text{-sets } G \subset \mathbb{R} (B \neq G \cap A)\}$

⁴Here $S(E) = ([0, 1] \times \{0\}) \cup (E \times \{1\})$ is considered as an ordered compactum with the lexicographical ordering.

is a club set C of levels of S such that for $\alpha \in C$, the value of τ_α is decided before the α^+ th level of S , where α^+ is the least member of C above α , in other words, every member of the α^+ th level decides the value of τ_α .

There is an S -name σ for a subset of ω_1 such that for all $\alpha \in C$, “ $\check{\alpha} \in \sigma$ ” is never decided until level $\alpha^+ + 1$. For instance, we can let σ be the set of pairs $(t, \check{\alpha}) \in T \times C$ such that $\alpha^+ \in \text{dom}(t)$ and $t(\alpha^+) \neq 0$. Then σ is forced to be a subset of C and t forces $\check{\alpha} \in \sigma$ if and only if α^+ is in the domain of t and $t(\alpha^+) \neq 0$. Then no real x from the ground model can code the realization of σ , since for each $\alpha \in C$, the statement “ $\check{x} \cap \tau_\alpha$ is infinite” is decided before the statement “ $\check{\alpha} \in \sigma$.” Since forcing with S adds no reals, we are done. \square

Corollary 6.6 (SA_{ω_1}). *Forcing with the Souslin tree S gives a model in which every compact space whose square is T_5 is metrizable.*

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