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Abstract. It is proved consistent with ZFC + GCH that for every Whitehead group A of infinite rank, there is a Whitehead group H_A such that $\mathrm{Ext}(H_A, A) \neq 0$. This is a strong generalization of the consistency of the existence of non-free Whitehead groups. A consequence is that it is undecidable in ZFC + GCH whether every $\mathbb{Z}$-module has a ${}^{\perp}\{\mathbb{Z}\}$-precover.

Let $\mathcal{F}$ be a class of R-modules of the form

$$^{\perp}\mathcal{C} = \{A : \mathrm{Ext}(A, C) = 0 \text{ for all } C \in \mathcal{C}\}$$

for some class $\mathcal{C}$.

A homomorphism $\phi \in \mathrm{Hom}(A, M)$ with $A \in \mathcal{F}$ is called an F-precover of M if the induced map $\mathrm{Hom}(A', A) \to \mathrm{Hom}(A', M)$ is surjective for all $A' \in \mathcal{F}$. (See [9] or [20].)

The first author and Jan Trlifaj proved [7] that a sufficient condition for every module M to have an $\mathcal{F}$-precover is that there is a module B such that $\mathcal{F}^{\perp} = \{B\}^{\perp}(= \{A : \mathrm{Ext}(B, A) = 0\})$. In [8], generalizing a method used by Enochs [1] to prove the Flat Cover Conjecture, it is proved that this sufficient condition holds whenever $\mathcal{C}$ is a class of pure-injective modules; moreover, for R a Dedekind domain, the sufficient condition holds whenever $\mathcal{C}$ is a class of cotorsion modules. The following is also proved in [8]:

Theorem 1. *It is consistent with ZFC + GCH that for any hereditary ring R and any R-module N, there is an R-module B such that $({}^{\perp}\{N\})^{\perp} = \{B\}^{\perp}$ and hence every R-module has a ${}^{\perp}\{N\}$-precover.*

This is a generalization of a result proved by the second author for the class $\mathcal{W}$ of all Whitehead groups ($= {}^{\perp}\{\mathbb{Z}\}$):

Theorem 2. *It is consistent with ZFC + GCH that $\mathcal{W}^{\perp} = \{B\}^{\perp}$ where B is any free abelian group.*

Proof. The second author proved that Gödel's Axiom of Constructibility (V = L) implies that $\mathcal{W}$ is exactly the class of free groups. (See [12] or [4].) Under this hypothesis (which implies GCH), $\mathcal{W}^{\perp}$ is the class of all groups; if we take B to be any free group, then $\{B\}^{\perp}$ is also the class of all groups. ∎

Our main result here is that the conclusions of Theorem 1 are not provable in ZFC + GCH for $N = \mathbb{Z} = R$:

Theorem 3. *It is consistent with ZFC + GCH that $\mathbb{Q}$ does not have a $\mathcal{W}$-precover.*

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An immediate consequence is:

Theorem 4. *It is consistent with ZFC + GCH that there is no abelian group B such that $\mathcal{W}^\perp = \{B\}^\perp$.*

Theorem 3 follows easily from the following:

Theorem 5. *It is consistent with ZFC + GCH that for every Whitehead group B there is an uncountable Whitehead group $G = G_B$ such that every homomorphism from G to B has finitely-generated range.*

Proof of Theorem 3 from Theorem 5. Suppose that $f : B \to \mathbb{Q}$ is a $\mathcal{W}$-precover of $\mathbb{Q}$. Let G be as in Theorem 5 for this B. Since $\mathbb{Q}$ is injective and G has infinite rank, there is a surjective homomorphism $g : G \to \mathbb{Q}$. But then clearly there is no $h : G \to B$ such that $f \circ h = g$. ∎

We get the hypothesis of Theorem 5 from the following:

Theorem 6. *Assume GCH. Suppose that for every Whitehead group A of infinite rank, there is a Whitehead group H_A of cardinality $\leq |A|^+$ such that* $\operatorname{Ext}(H_A, A) \neq 0$. *Then for every Whitehead group B there is an uncountable Whitehead group G such that every homomorphism from G to B has finitely-generated range.*

Proof. Let $\lambda = \mu^+$ where $\mu > |B| + \aleph_1$. Then $\diamondsuit_\lambda$ holds, by GCH (cf. [11], [15]); we will use it to construct the group structure on a set G of size λ. We can write $G = \bigcup_{\nu<\lambda} G_\nu$ as the union of a continuous chain of sets such that for all $\nu < \lambda$, $|G_{\nu+1} - G_\nu| = \mu$. Now $\diamondsuit_\lambda$ gives us a family $\{h_\nu : \nu \in \lambda\}$ of functions $h_\nu : G_\nu \to B$ such that for every function $f : G \to B$, $\{\nu \in \lambda : f \restriction G_\nu = h_\nu\}$ is stationary.

Suppose that the group structure on G_ν has been defined and consider h_ν; if the range of h_ν is of finite rank, define the group structure on $G_{\nu+1}$ in any way which extends that on G_ν. Otherwise, let A be the range of h_ν and let H_A be as in the hypothesis. Without loss of generality, $|H_A| = \mu$; write $H_A = F/K$ where F is a free group of rank μ. By a standard homological argument, there is a homomorphism $\psi : K \to A$ which does not extend to a homomorphism $\varphi : F \to A$. Since K is free and $h_\nu : G_\nu \to B$ is onto A, there is a homomorphism $\theta : K \to G_\nu$ such that $h_\nu \circ \theta = \psi$. Now form the pushout

$$\begin{array}{ccc} F & \to & G_{\nu+1} \\ \uparrow & & \uparrow \\ K & \stackrel{\theta}{\to} & G_\nu \end{array}$$

to define the group structure on $G_{\nu+1}$ (cf. [7, proof of Theorem 2]). Then $G_{\nu+1}/G_\nu \cong F/K \cong H_A$ so it is Whitehead. Moreover, h_ν does not extend to a homomorphism from $G_{\nu+1}$ into A, else ψ extends to a homomorphism on F. This completes the definition of G. Notice that G is a Whitehead group since all quotients $G_{\nu+1}/G_\nu$ are isomorphic to F/K and hence Whitehead (cf. [7, Lemma 1]).

Now given any homomorphism $f : G \to B$, let $A \subseteq B$ be the range of f. Since $|A| < |G| = \lambda$, $\{\nu \in \lambda : f[G_\nu] = A\}$ is a club in λ; hence there exists $\nu \in \lambda$ such that $f \restriction G_\nu = h_\nu$ and the range of h_ν is A. If A is of infinite rank, we have constructed $G_{\nu+1}$ so that $f \restriction G_\nu$ does not extend to $G_{\nu+1}$, which is a contradiction. So we must conclude that the range of f is of finite rank. ∎

Now our main task is to show that there is a model of ZFC + GCH where the hypothesis of Theorem 6 holds. As a warm-up exercise, however, we will begin in the next section with a direct proof of Theorem 4; this is equivalent to the consistency of a weaker assumption than the hypothesis of Theorem 6.

1. $\mathcal{W}$ IS NOT COGENERATED BY A SET

Theorem 4 is equivalent to the statement that it is consistent with ZFC + GCH that for every W-group B we can find a W-group $A \in \{B\}^{\perp}$ such that there is a W-group H_A with $\text{Ext}(H_A, A) \neq 0$. The proof will use the following consequence of Theorem 2 of [7]:

Theorem 7. *Let μ be a cardinal $> \kappa$ such that $\mu^{\kappa} = \kappa$ and let B be a group of cardinality $\leq \kappa$. Then there is a group $A \in \{B\}^{\perp}$ such that $A = \cup_{\nu<\mu} A'_{\nu}$ (continuous), $A'_0 = 0$, and such that for all $\nu < \sigma$, $A_{\nu+1}/A'_{\nu}$ is isomorphic to B.* ∎

Proof of Theorem 4. We will use the fact that the following principle is consistent with ZFC + GCH (cf. [6]):

> **(UP)** For every cardinal σ of the form τ^{+} where τ is singular of cofinality ω there is a stationary subset S of σ consisting of limit ordinals of cofinality ω and a ladder system $\bar{\zeta} = \{\zeta_\delta : \delta \in S\}$ which has the ω-uniformization property, that is, for every family $\{c_\delta : \delta \in S\}$ of functions $c_\delta : \omega \to \omega$, there is a function $h : \sigma \to \omega$ such that for every $\delta \in S$, $h(\zeta_\delta(n)) = c_\delta(n)$ for almost all $n \in \omega$.

We work in a model of GCH plus UP. Given $B \in \mathcal{W}$, let $\kappa = |B|$ and let μ be a singular cardinal of cofinality $\sigma > \kappa$ such that σ is the successor of a singular cardinal of cofinality ω. Then $\mu^{\kappa} = \kappa$. Let $A = \cup_{\nu<\sigma} A'_{\nu}$ be as in Theorem 7 for this B and μ. Choose a strictly increasing continuous function $\xi : \sigma \to \mu$ whose range in cofinal in μ and let $A_\nu = A'_{\xi(\nu)}$. Let $\bar{\zeta} = \{\zeta_\delta : \delta \in S\}$ be as in (UP).

Let $H_A = F/K$ where F is the free group on symbols $\{y_{\delta,n} : \delta \in S, n \in \omega\} \cup \{x_j : j < \sigma\}$ and K is the subgroup with basis $\{w_{\delta,n} : \delta \in S, n \in \omega\}$ where

$$w_{\delta,n} = 2y_{\delta,n+1} - y_{\delta,n} + x_{\zeta_\delta(n)}. \tag{1}$$

Then H_A is a group of cardinality σ and the ω-uniformization property of $\bar{\zeta}$ implies that H_A is a Whitehead group (see [5, XII.3] or [19]). We will show that $\text{Ext}(H_A, A) \neq 0$ by defining $\psi : K \to A$ which does not extend to a homomorphism $\varphi : F \to A$.

For all $\nu < \mu$, A/A_ν is a W-group since B is, and hence A/A_ν is strongly $\aleph_1$-free, since CH holds; thus it has a pure free subgroup C/A_ν of rank ω with basis $\{t_{\nu,n} + A_\delta : n \in \omega\}$ such that A/C is $\aleph_1$-free. Then $a_\delta = \Sigma_{n\in\omega} 2^n (t_{\nu,n} + A_\delta)$ is in the 2-adic completion of A/A_δ but not in A/A_δ. Define $\psi : K \to A$ such that $\psi(w_{\delta,n}) = t_{\delta,n}$ for all $\delta \in S$, $n \in \omega$. Suppose, to obtain a contradiction, that ψ extends to a homomorphism $\varphi : F \to A$. The set of $\delta < \sigma$ such that $\varphi(x_j) \in A_\delta$ for all $j < \delta$ is a club, C, in σ, so there exists $\delta \in S \cap C$. We will contradict the choice of a_δ for this δ.

We work in A/A_δ. Let $c_n = \varphi(y_{\delta,n}) + A_\delta$. Then by applying φ to the equations (1) and since $\varphi(x_j) \in A_\delta$ for all $j < \delta$ we have that for all $n \in \omega$,

$$t_{\delta,n} + A_\delta = 2c_{n+1} + c_n.$$

It follows that $a_\delta = c_0$ is in A/A_δ, a contradiction. ∎

This completes the proof of the weaker Theorem 4. In Theorem 6, we are not able to choose the Whitehead group A, but must find an H_A for *every* A. In the next section we discuss ways to insure that $\mathrm{Ext}(H_A, A)$ is non-zero, and in the following section we deal with how to make H_A a Whitehead group, and then finish the proof of the main theorem.

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2. How to make Ext not vanish

We begin by proving some general properties of decompositions of Whitehead groups assuming GCH. We use the result of Gregory and Shelah (cf. [11], [15]) that GCH implies $\diamondsuit_\lambda$ for every successor cardinal $\lambda > \aleph_1$, and the result of Devlin and Shelah [3] that CH implies weak diamond, $\Phi_{\aleph_1}$, at $\aleph_1$. We will also make repeated use of the fact (cf. [12], [3], [5, Chap XII]) that if $A = \bigcup_{\alpha<\lambda} A_\alpha$ is a λ filtration of a group of cardinality λ and if $\Phi_\lambda(E)$ holds where $E = \{\alpha \in \lambda : \exists \beta > \alpha \text{ s.t. } A_\beta/A_\alpha$ is not Whitehead$\}$, then A is not a Whitehead group.

Lemma 8. *Let A be a Whitehead group of cardinality $\lambda = \mu^+$ and write $A = \bigcup_{\alpha<\lambda} A_\alpha$ as the continuous union of a chain of subgroups of cardinality μ. Let $S(A) \stackrel{def}{=} \{\alpha \in \lambda : A_\tau/A_\alpha$ is Whitehead for all $\tau > \alpha\}$. If $\Phi_\lambda(Y)$ holds for some subset Y of λ, then $Y \cap S(A)$ is stationary. In particular, assuming GCH, $S(A)$ is stationary.*

Proof. Suppose $Y \cap S(A)$ is not stationary in λ, and let C be a club in its complement; then we can define a continuous increasing function $f : \lambda \to C$ such that for all $\alpha \in \lambda$, if $f(\alpha) \in Y$, then $A_{f(\alpha+1)}/A_{f(\alpha)}$ is not Whitehead. But then $\Phi_\lambda(Y \cap \mathrm{im}(f))$ holds and implies that $A = \bigcup_{\alpha\in\lambda} A_{f(\alpha)}$ is not Whitehead. ∎

We can say, for short, that A/A_α is *locally Whitehead* when $\alpha \in S(A)$, since every subgroup of A/A_α of cardinality $< \lambda$ is Whitehead.

Lemma 9. *Assume GCH. Let A be a Whitehead group of cardinality μ (possibly a singular cardinal). Then we can write $A = \bigcup_{\nu<\mu} A_\nu$ as the continuous union of a chain of subgroups of cardinality $< \mu$ such that for all $\nu < \mu$, $A/A_{\nu+1}$ is $\aleph_1$-free.*

Proof. If suffices to show that every subgroup X of A of cardinality $\kappa < \mu$ is contained in a subgroup N of cardinality κ such that N'/N is free whenever $N \subseteq N' \subseteq A$ and N'/N is countable. But if X is a counterexample, then we can build a chain $\{N_\alpha : \alpha < \kappa^+\}$ such that $N_0 = X$ and for all $\alpha < \kappa^+$, $N_{\alpha+1}/N_\alpha$ is countable and not free, and hence is not Whitehead. We obtain a contradiction since then Φ_{κ^+} implies that $\bigcup_{\alpha<\kappa^+} N_\alpha$ is not Whitehead. ∎

We now give sufficient conditions for $\mathrm{Ext}(H, A)$ to be non-zero, when H is given by a relative simple set of relations defined using ladder systems (see the definition below). The analysis will be divided into cases, depending on whether the cardinality of A is singular, the successor of a regular cardinal, or the successor of a singular cardinal.

The following concrete description of a group is in the spirit of the general constructions in, for example, [19] or [5, XII.3.4] but is a little more complicated since it is "two step": involving a system of ladders of length $\mathrm{cf}(\mu)$ and another system of ladders of length ω (if $\mathrm{cf}(\mu) > \aleph_0$).

Definition 10. Let μ be a cardinal of cofinality σ ($\leq \mu$). Let S be a subset of $\lambda = \mu^+$ consisting of ordinals of cofinality σ and $\bar{\eta} = \{\eta_\delta : \delta \in S\}$ a ladder system on S, that is, a family of functions $\eta_\delta : \sigma \to \delta$ which are strictly increasing and cofinal. If $\sigma > \aleph_0$, let E be a stationary subset of σ consisting of limit ordinals of cofinality ω and let $\bar{\zeta} = \{\zeta_\nu : \nu \in E\}$ be a ladder system on E. We will say that H is *the group built on* $\bar{\eta}$ *and* $\bar{\zeta}$ if $H \cong F/K$ where F is the free group on symbols $\{y_{\delta,\nu,n} : \delta \in S,\, \nu \in E,\, n \in \omega\} \cup \{z_{\delta,j} : \delta \in S,\, j \in \sigma\} \cup \{x_\beta : \beta \in \lambda\}$ and K is the subgroup with basis $\{w_{\delta,\nu,n} : \delta \in S,\, \nu \in E,\, n \in \omega\}$ where

$$w_{\delta,\nu,n} = 2y_{\delta,\nu,n+1} - y_{\delta,\nu,n} - z_{\delta,\zeta_\nu(n)} + x_{\eta_\delta(\nu+n)}. \tag{2}$$

(If $\sigma = \aleph_0$, let $E = \{0\}$ and omit $\bar{\zeta}$ and the $z_{\delta,j}$.) For future reference, let F_α be the subgroup of F generated by $\{y_{\delta,\nu,n} : \delta \in S \cap \alpha,\, \nu \in E,\, n \in \omega\} \cup \{z_{\delta,j} : \delta \in S \cap \alpha,\, j < \sigma\} \cup \{x_\beta : \beta < \alpha\}$ and for $\alpha \in S$ and $\tau < \sigma$ let $F_{\alpha,\tau}$ be the subgroup generated by $\{z_{\alpha,j} : j < \tau\}$.

When the cardinality of A is singular, we will use a special case of a recent result of the second author [17]. For convenience, we give the statement and proof of this "very weak diamond" result here.

Lemma 11. *Assume GCH. Let μ be a singular cardinal and let $\sigma = \mathrm{cof}(\mu) < \mu$ and $\lambda = \mu^+$. Suppose that S is a stationary subset of λ consisting of ordinals of cofinality σ and $\{\eta_\delta : \delta \in S\}$ is a ladder system on S. Then for each $\delta \in S$ there is a sequence of sets $D^\delta = \langle D^\delta_\nu : \nu < \sigma\rangle$ such that*

(a) for all $\delta \in S$ and $\nu \in \sigma$, $D^\delta_\nu \subseteq \lambda$, $\sup(D^\delta_\nu) < \delta$ and $|D^\delta_\nu| < \mu$; and

(b) for every function $h : \lambda \to \lambda$, $\{\delta \in S : h(\eta_\delta(\nu)) \in D^\delta_\nu$ for all $\nu \in \sigma\}$ is stationary in λ.

Proof. Fix $\delta \in S$. Let $\langle b^\delta_\nu : \nu < \sigma\rangle$ be an increasing continuous union of subsets of δ whose union is δ and such that $\sup(b^\delta_\nu) < \delta$ and $\mathrm{card}(b^\delta_\nu) < \mu$. Let $\theta = 2^\sigma = \sigma^+ (< \mu)$ and let $\langle g_i : i < \theta\rangle$ be a list of all functions from σ to σ. Also let $\langle f_\gamma : \gamma < \lambda\rangle$ list all functions from θ to λ $(= 2^\mu = \lambda^\theta)$; without loss of generality, $f_\gamma(i) < \gamma$ for all $i \in \theta$. For each $i \in \theta$ and $\nu \in \theta$, define $D^{i,\delta}_\nu = \{f_\gamma(i) : \gamma \in b^\delta_{g_i(\nu)}\}$.

We claim that for some $i \in \theta$, the sets $\{D^{i,\delta} = \langle D^{i,\delta}_\nu : \nu < \sigma\rangle : \delta \in S\}$ will work in (b). Assuming the contrary, for each $i \in \theta$, let $h_i : \lambda \to \lambda$ be a counterexample, i.e., there is a club C_i in λ such that for each $\delta \in C_i \cap S$, there is $\nu \in \sigma$ such that $h_i(\eta_\delta(\nu)) \notin D^{i,\delta}_\nu$.

For each $\alpha \in \lambda$, there is $h(\alpha) \in \lambda$ such that for all $i \in \theta$, $h_i(\alpha) = f_{h(\alpha)}(i)$. There exists $\delta_* \in \bigcap_{i\in\theta} C_i \cap S$ such that for all $\alpha < \delta_*$, $h(\alpha) \in \delta_*$. Denote $h(\eta_{\delta_*}(\nu))$ by γ_ν. There exists $i_* \in \theta$ such that for all $\nu < \sigma$,

$$g_{i_*}(\nu) = \min\{j < \sigma : \gamma_\nu \in b^{\delta_*}_j\}.$$

(Note that the right-hand side exists since $\delta_* = \cup_{j<\sigma} b^{\delta_*}_j$ and $\gamma_\nu \in \delta_*$.) Thus

$$\gamma_\nu \in b^{\delta_*}_{g_{i_*}(\nu)}.$$

But then, (letting $\alpha = \eta_{\delta_*}(\nu)$ in the definition of h),

$$h_{i_*}(\eta_{\delta_*}(\nu)) = f_{h(\eta_{\delta_*}(\nu))}(i_*) = f_{\gamma_\nu}(i_*) \in D^{i_*,\delta_*}_\nu.$$

Since this holds for all $\nu \in \sigma$, the fact that h_{i_*} is a counterexample implies that $\delta_* \notin C_{i_*} \cap S$. But this contradicts the choice of δ_*. ∎

Theorem 12. *Assume GCH. Let μ be a singular cardinal of cofinality σ. If H is a group of cardinality $\lambda = \mu^+$ built on $\bar{\eta}$ and $\bar{\zeta}$ as in Definition 10 and A is a Whitehead group of cardinality μ, then* $\mathrm{Ext}(H, A) \neq 0$.

Proof. Let the sets $\{D^\delta = \langle D^\delta_\nu : \nu \in \sigma\rangle : \delta \in S\}$ be as in Lemma 11 for this ladder system. Write $A = \bigcup_{\nu<\mu} A_\nu$ as in Lemma 9. Without loss of generality we can assume that the universe of A is μ.

We claim that for all $\beta < \mu$, the 2-adic completion of A/A_β has rank $\geq \mu$ over A/A_β. For notational convenience we will prove the case $\beta = 0$, but the argument is the same in general using the decomposition $A/A_\beta = \bigcup_{\beta\leq\alpha<\mu} A_\alpha/A_\beta$. Since $A_{\alpha+1}/A_\alpha$ is $\aleph_1$-free and non-zero, there are $s^\alpha_n \in A_{\alpha+1}$ such that the element $\Sigma_{n\in\omega}2^n(s^\alpha_n + A_\alpha)$ of the 2-adic completion of $A_{\alpha+1}/A_\alpha$ is not in $A_{\alpha+1}/A_\alpha$. We claim that the elements $\{\Sigma_{n\in\omega}2^n s^\alpha_n : \alpha \in \mu\}$ of the 2-adic completion of A are linearly independent over A. Suppose not, and let

$$\Sigma^m_{i=1} k_i(\Sigma_{n\in\omega}2^n s^{\alpha(i)}_n) = a$$

be a counterexample; so $a \in A$; $k_i \in \mathbb{Z}-\{0\}$; and $\alpha(1) < \alpha(2) < ... < \alpha(m) < \mu$. Let $\gamma = \alpha(m)$ and $k = k_\gamma$. We claim that the element $k\Sigma_{n\in\omega}2^n(s^\gamma_n + A_\gamma)$ of the 2-adic completion of $A_{\gamma+1}/A_\gamma$ belongs to $A_{\gamma+1}/A_\gamma$ which is a contradiction of the choice of the s^γ_n. Since $A/A_{\gamma+1}$ is $\aleph_1$-free, we can write $\langle A_{\gamma+1}, a\rangle_* = A_{\gamma+1} \oplus C$ for some C, and let a' be the projection of a on the first factor. For every $r \in \omega$, 2^{r+1} divides $a- \Sigma^m_{i=1}k_i(\Sigma^r_{n=0}2^n s^{\alpha(i)}_n)$ in A and hence 2^{r+1} divides $a'- \Sigma^m_{i=1}k_i(\Sigma^r_{n=0}2^n s^{\alpha(i)}_n)$ in $A_{\gamma+1}$. But then 2^{r+1} divides $(a' + A_\gamma)- k\Sigma^r_{n=0}2^n(s^\gamma_n + A_\gamma)$ in $A_{\gamma+1}/A_\gamma$; since this holds for all $r \in \omega$, $k\Sigma_{n\in\omega}2^n(s^\gamma_n + A_\gamma) = a' + A_\gamma$, and we have a contradiction.

Choose a strictly increasing continuous function $\xi : \sigma \to \mu$ whose range is cofinal in μ. For each $\delta \in S$ and $\nu \in E$, there is an element $a_{\delta,\nu} = \Sigma_{n\in\omega}2^n(a(\delta,\nu,n) + A_{\xi(\nu)+1})$ in the 2-adic completion of $A/A_{\xi(\nu)+1}$ which is not in the subgroup generated by $A/A_{\xi(\nu)+1}$ and the 2-adic completion of $\{d + A_{\xi(\nu)+1} : d \in D^\delta_\nu \cap A\}$. (Note that the latter has cardinality $< \mu$ since $|D^\delta_\nu|^{\aleph_0} < \mu$ by the GCH.)

Now define $\psi : K \to A$ such that $\psi(w_{\delta,\nu,n}) = a(\delta,\nu,n)$. We claim that ψ does not extend to a homomorphism $\varphi : F \to A$. Suppose, to the contrary, that it does. Then by Lemma 11, there is $\delta \in S$ such that $\varphi(x_{\eta_\delta(\nu)}) \in D^\delta_\nu$ for all $\nu \in \sigma$. Now there exists $\nu \in E$ such that $\varphi(z_{\delta,j}) \in A_{\xi(\nu)}$ for all $j < \nu$. We will contradict the choice of $a_{\delta,\nu}$ for this δ and ν.

We work in $A/A_{\xi(\nu)+1}$. Let $c_n = \varphi(y_{\delta,\nu,n})+A_{\xi(\nu)+1}$, $d_n = \varphi(x_{\eta_\delta(\nu+n)})+A_{\xi(\nu)+1}$. Then by applying φ to the equations (1) and since $\varphi(z_{\delta,j}) \in A_{\xi(\nu)}$ for all $j < \nu$ we have that for all $n \in \omega$,

$$a(\delta,\nu,n) + A_{\xi(\nu)+1} = 2c_{n+1} - c_n + d_n.$$

It follows that $a_{\delta,\nu} = c_0 + \Sigma_{n\in\omega}2^n d_n$ is in the subgroup generated by $A/A_{\xi(\nu)+1}$ and the 2-adic completion of $\{d + A_{\xi(\nu)+1} : d \in D^\delta_\nu \cap A\}$, which contradicts the choice of $a_{\delta,\nu}$. ∎

We now turn to the cases when the cardinality of A is a successor cardinal. Though the two arguments could be combined into one, following the argument in Theorem 15, we prefer to introduce the method with the somewhat simpler argument for the successor of regular case. The following lemma is easy to confirm:

Lemma 13. *Suppose that L' is a free subgroup of L such that L/L' is $\aleph_1$-free. If $\{t_n : n \in \omega\}$ is a basis of a summand of L', then $\sum_{n\in\omega} 2^n t_n$ is an element of the 2-adic completion of L which does not belong to L. In other words, the system of equations*

$$2y_{n+1} = y_n - t_n$$

in the unknowns y_n $(n \in \omega)$ does not have a solution in L. ∎

Theorem 14. *Assume GCH. Let $\lambda = \mu^+$ where μ is a regular cardinal. Suppose H is built on $\bar{\eta} = \{\eta_\delta : \delta \in S\}$ and $\bar{\zeta} = \{\zeta_\nu : \nu \in E\}$ as in Definition 10. Suppose also, for $\mu > \aleph_0$, that $\diamondsuit_\mu(E')$ holds for all stationary subsets E' of E. If A is a Whitehead group of cardinality $\lambda = \mu^+$, then* $\operatorname{Ext}(H, A) \neq 0$.

Proof. Let $A = \bigcup_{\alpha<\lambda} A_\alpha$ and $S(A)$ be as in Lemma 8. Note that we make no assumption about the relation of S and $S(A)$; maybe $S \cap S(A) = \emptyset$. Without loss of generality, for all $\delta \in S(A)$, $A_{\delta+1}/A_\delta$ is Whitehead of rank μ and $A/A_{\delta+1}$ is locally Whitehead. Assume $\mu > \aleph_0$; the proof for $\aleph_0$ is simpler. For each $\alpha < \lambda$, write A_α as the union of a continuous chain of subgroups of cardinality $< \mu$: $A_\alpha = \bigcup_{\nu<\mu} B_{\alpha,\nu}$. Thus $A_{\delta+1}/A_\delta = \bigcup_{\nu<\mu}(A_\delta + B_{\delta+1,\nu})/A_\delta$; for $\delta \in S(A)$, since $\diamondsuit_\mu(E)$ holds, we can assume that the set of $\nu \in E$ such that $A_{\delta+1}/(A_\delta + B_{\delta+1,\nu})$ is locally Whitehead is stationary; for such ν, $A_{\delta+1}/A_\delta + B_{\delta+1,\nu}$ is then strongly $\aleph_1$-free since CH holds (cf. [2]). Thus for ν in a stationary subset E_δ of E we can assume that $A_\delta + B_{\delta+1,\nu+1}/A_\delta + B_{\delta+1,\nu}$ is free of rank $\aleph_0$ and $A_{\delta+1}/A_\delta + B_{\delta+1,\nu+1}$ is $\aleph_1$-free. Say $\{t_{\delta,\nu,n} + A_\delta + B_{\delta+1,\nu} : n \in \omega\}$ is a basis of $A_\delta + B_{\delta+1,\nu+1}/A_\delta + B_{\delta+1,\nu}$.

For each $\delta_1 \in S$, let δ_1^+ be the least member of $S(A)$ which is $\geq \delta_1$. Define

$$\psi(w_{\delta_1,\nu,n}) = t_{\delta_1^+,\nu,n}$$

for all $n \in \omega$ if $\nu \in E_{\delta_1^+}$. Define ψ arbitrarily otherwise. We claim that ψ does not extend to $\varphi : F \to A$. Suppose to the contrary that it does. Let $M = \varphi[F]$, $M_\alpha = \varphi[F_\alpha]$, $M_{\alpha,\tau} = \varphi[F_{\alpha,\tau}]$. Then there is a club C in λ such that for $\alpha \in C$, $M_\alpha \subseteq A_\alpha$. Fix δ_1 in $C \cap S$. Let δ denote δ_1^+ and choose $\gamma \in C$ such that $\gamma > \delta$. There is a club C' in μ such that for $\nu \in C'$, $M_{\delta_1,\nu} \subseteq B_{\gamma,\nu}$ and $A_{\delta+1} \cap B_{\gamma,\nu} \subseteq B_{\delta+1,\nu}$. Since $\diamondsuit_\mu(E_\delta)$ holds and A_γ/A_δ is Whitehead, there is, by Lemma 8, $\nu \in E_\delta \cap C'$ such that $A_\gamma/(A_{\delta+1} + B_{\gamma,\nu})$ is locally Whitehead, and hence $\aleph_1$-free. We will obtain a contradiction of Lemma 13 with $L = A_\gamma/(A_\delta + B_{\gamma,\nu})$ and $L' = (B_{\delta+1,\nu+1} + A_\delta + B_{\gamma,\nu})/(A_\delta + B_{\gamma,\nu})$ and $t_n = t_{\delta,\nu,n} + A_\delta + B_{\gamma,\nu}$. Notice that modulo $A_\delta + B_{\gamma,\nu}$ we have

$$2\varphi(y_{\delta_1,\nu,n+1}) = \varphi(y_{\delta_1,\nu,n}) - t_{\delta,\nu,n}$$

for all $n \in \omega$ since $\varphi(x_{\eta_{\delta_1}(\nu+n)}) \in A_{\delta_1} \subseteq A_\delta$ and $\varphi(z_{\delta_1,\zeta_\nu(n)}) \in M_{\delta_1,\nu} \subseteq B_{\gamma,\nu}$. Moreover, $\{t_n : n \in \omega\}$ is a basis of a summand of L' since L' is naturally isomorphic to $A_\delta + B_{\delta+1,\nu+1}/A_\delta + (B_{\gamma,\nu} \cap (A_\delta + B_{\delta+1,\nu+1}))$ and the latter has a natural epimorphism onto $A_\delta + B_{\delta+1,\nu+1}/A_\delta + B_{\delta+1,\nu}$ which is free on the basis $\{t_{\delta,\nu,n} + A_\delta + B_{\delta+1,\nu} : n \in \omega\}$. It remains to show that L/L' is $\aleph_1$-free. Now

$$0 \to (A_{\delta+1} + B_{\gamma,\nu})/(B_{\delta+1,\nu+1} + A_\delta + B_{\gamma,\nu}) \to L/L' \to A_\gamma/(A_{\delta+1} + B_{\gamma,\nu}) \to 0$$

is exact and $A_\gamma/(A_{\delta+1} + B_{\gamma,\nu})$ is $\aleph_1$-free by choice of ν, so it suffices to show that $(A_{\delta+1} + B_{\gamma,\nu})/(B_{\delta+1,\nu+1} + A_\delta + B_{\gamma,\nu})$ is $\aleph_1$-free. But this is isomorphic to $A_{\delta+1}/((A_\delta + B_{\delta+1,\nu+1}) + (A_{\delta+1} \cap B_{\gamma,\nu}))$, which (since $A_{\delta+1} \cap B_{\gamma,\nu} \subseteq B_{\delta+1,\nu} \subseteq B_{\delta+1,\nu+1}$) equals $A_{\delta+1}/(A_\delta + B_{\delta+1,\nu+1})$, which was chosen $\aleph_1$-free. ∎

For the general case we need to use elementary submodels (cf. [5, §VI.2]).

Theorem 15. *Assume GCH. Let $\lambda = \mu^+$ where μ is a (singular) cardinal of cofinality σ. Suppose H is built on $\bar{\eta} = \{\eta_\delta : \delta \in S\}$ and $\bar{\zeta} = \{\zeta_\nu : \nu \in E\}$ as in Definition 10. Suppose also that $\diamondsuit_\lambda(Y)$ holds for some subset Y of λ consisting of* *limit ordinals of cofinality σ and that, if $\sigma > \aleph_0$, $\diamondsuit_\sigma(E)$ holds. If A is a Whitehead group of cardinality $\lambda = \mu^+$, then* $\operatorname{Ext}(H, A) \neq 0$.

PROOF. Without loss of generality, for all $\delta \in S(A)$, $A_{\delta+1}/A_\delta$ is Whitehead of rank μ. For each $\delta \in S$, choose a strictly increasing sequence $\langle \xi_{\delta,\nu} : \nu < \sigma \rangle$ of elements of $S(A)$ such that $\xi_{\delta,0} \geq \delta+1$ and whose limit, denoted $\xi_{\delta,\sigma}$, belongs to $S(A)$. This is possible because, by Lemma 8, $\overline{Y \cap S(A)}$ is stationary so we can choose $\xi_{\delta,\sigma}$ to be an element of $Y \cap S(A) \cap \overline{(S(A) \cap (\delta,\lambda))}$ where $\overline{(S(A) \cap (\delta,\lambda))}$ is the closure of $\{\alpha \in S(A) : \alpha > \delta\}$. Let $B_{\delta+1,\nu} = A_{\xi_{\delta,\nu}}$. (Note the difference from the last proof.) We can then modify the sequence so that $B_{\delta+1,\nu+1}/B_{\delta+1,\nu}$ is free on a countable set $\{t_{\delta,\nu,n} + B_{\delta+1,\nu}\}$ and $A/B_{\delta+1,\nu+1}$ is $\aleph_1$-free when $\nu \in E$. (For $\nu \in E$ we drop the requirement that $\xi_{\delta,\nu+1} \in S(A)$.)

For each $\delta_1 \in S$, let δ_1^+ be the least member of $S(A)$ which is $\geq \delta_1$. Define

$$\psi(w_{\delta_1,\nu,n}) = t_{\delta_1^+,\nu,n}$$

for all $n \in \omega$, $\nu \in E$. We claim that ψ does not extend to $\varphi : F \to A$. Suppose to the contrary that it does. As before, let $M = \varphi[F]$, $M_\alpha = \varphi[F_\alpha]$, $M_{\alpha,\tau} = \varphi[F_{\alpha,\tau}]$ and let C be a club such that for $\alpha \in C$, $M_\alpha \subseteq A_\alpha$. Fix δ_1 in $C \cap S$. Let δ be δ_1^+ and choose $\gamma \in C$ such that $\gamma > \delta$.

Let $N = \bigcup_{\nu<\sigma} N_\nu$ be the continuous union of a chain of elementary submodels of $H(\chi)$ for large enough χ such that each N_ν has cardinality $< \sigma$, $N_\nu \in N_{\nu+1}$ and such that δ, σ, A, $\{\varphi(z_{\delta_1,\nu}) : \nu < \sigma\}$, $\{\varphi(x_{\eta_{\delta_1}(\nu+n)}) : \nu < \sigma\}$ (for each $n \in \omega$), $\{t_{\delta,\nu,n} : \nu < \sigma, n \in \omega\}$ and $\{\xi_{\delta,\nu} : \nu \leq \sigma\}$ all belong to N_0 and

$$\{\varphi(z_{\delta_1,j}) : j < \sigma\} \cup \{\varphi(x_{\eta_{\delta_1}(\nu)}) : \nu < \sigma\} \cup \{t_{\delta,\nu,n} : \nu < \sigma, n \in \omega\} \cup \sigma \subseteq N.$$

Moreover, by intersecting with a club, we can assume that for all ν, $N_\nu \cap \sigma = \nu$ and $N_\nu \cap B_{\delta+1,\sigma} \subseteq B_{\delta+1,\nu}$ and hence $\{\xi_{\delta,j} : j < \nu\}$, $\{\varphi(z_{\delta_1,j}) : j < \nu\}$, $\{t_{\delta,j,n} : j < \nu, n \in \omega\}$, and $\{\varphi(x_{\eta_{\delta_1}(j+n)}) : j < \nu\}$ (for all $n \in \omega$) are all subsets of N_ν. We claim that there is a $\nu \in E$ such that $A/(B_{\delta+1,\sigma} + (N_\nu \cap A))$ is Whitehead, and hence $\aleph_1$-free. Assuming this for the moment, we show how to obtain a contradiction of Lemma 13 with $L = (N \cap A)/((N \cap A_\delta) + (N_\nu \cap A))$, $L' = ((N \cap B_{\delta+1,\nu+1}) + (N_\nu \cap A))/((N \cap A_\delta) + (N_\nu \cap A))$ and $t_n = t_{\delta,\nu,n} + ((N \cap A_\delta) + (N_\nu \cap A))$. Notice that for all $n \in \omega$, $\varphi(x_{\eta_{\delta_1}(\nu+n)}) \in (N \cap A_\delta)$ and $\varphi(z_{\delta_1,\zeta_\nu(n)}) \in N_\nu$. Moreover, $\{t_n : n \in \omega\}$ is a basis of a summand of L' because L' is naturally isomorphic to $(N \cap B_{\delta+1,\nu+1})/(N \cap A_\delta) + (N_\nu \cap B_{\delta+1,\nu})$ and the latter has epimorphic image $(N \cap B_{\delta+1,\nu+1})/(N \cap B_{\delta+1,\nu})$ which is free on the basis $\{t_{\delta,\nu,n} + (N \cap B_{\delta+1,\nu}) : n \in \omega\}$ by choice of N. To see that L/L' is $\aleph_1$-free, use the short exact sequence

$$0 \to ((N \cap B_{\delta+1,\sigma}) + (N_\nu \cap A))/((N \cap B_{\delta+1,\nu+1}) + (N_\nu \cap A)) \to$$
$$L/L' \to (N \cap A)/((N \cap B_{\delta+1,\sigma}) + (N_\nu \cap A)) \to 0.$$

The last term is $\aleph_1$-free by choice of ν and since N is an elementary submodel of $H(\chi)$. Moreover, $((N \cap B_{\delta+1,\sigma}) + (N_\nu \cap A))/((N \cap B_{\delta+1,\nu+1}) + (N_\nu \cap A))$ is isomorphic to $(N \cap B_{\delta+1,\sigma})/(N \cap B_{\delta+1,\nu+1})$ (since $N_\nu \cap B_{\delta+1,\sigma} \subseteq B_{\delta+1,\nu}$) and thus is $\aleph_1$-free since $A/B_{\delta+1,\nu+1}$ is $\aleph_1$-free.

It remains to show that there is a $\nu \in E$ such that $A/(B_{\delta+1,\sigma} + (N_\nu \cap A))$ is Whitehead. If not, then for all $\nu \in E$, $(B_{\delta+1,\sigma} + (N_{\nu+1} \cap A))/(B_{\delta+1,\sigma} + (N_\nu \cap A))$ is not Whitehead, since A, $B_{\delta+1,\sigma}$ and N_ν belong to the elementary submodels $N_{\nu+1}$ and N. But then $\diamondsuit_\sigma(E)$ implies that $\bigcup_{\nu<\sigma}(B_{\delta+1,\sigma} + (N_\nu \cap A))/B_{\delta+1,\sigma}$ is a group of cardinality σ which is not a Whitehead group, contradicting the fact that $A/B_{\delta+1,\sigma} = A/A_{\xi_{\delta,\sigma}}$ is locally Whitehead. ∎

3. Whitehead groups by uniformization

We present a special case of a theorem of Shelah and Strüngmann [18].

Theorem 16. *Suppose that H is built from $\bar{\eta}$ and $\bar{\zeta}$ as in Definition 10 and that E is a non-reflecting subset of σ. Then H is a Whitehead group if $\bar{\eta}$ satisfies ω-uniformization, that is, given any functions $c_\delta : \sigma \to \omega$ for $\delta \in S$, there is a pair (f, f^*) where $f : \lambda \to \omega$ and $f^* : S \to \sigma$ such that for all $\delta \in S$, $f(\eta_\delta(\nu)) = c_\delta(\nu)$ whenever $f^*(\delta) \leq \nu < \sigma$.*

Proof. We assume $\sigma > \aleph_0$ since this is known otherwise (cf. [13], [19]). If F and K are as in Definition 10, it suffices to show that every homomorphism $\psi : K \to \mathbb{Z}$ extends to a homomorphism $\varphi : F \to \mathbb{Z}$. Given ψ, define $c_\delta(\nu + n) = \psi(w_{\delta,\nu,n})$ for $\nu \in E$, and arbitrary otherwise. Let (f, f^*) be the uniformizing pair. Define $\varphi(x_\beta) = f(\beta)$. For each $\delta \in S$ we must still define $\varphi(y_{\delta,\nu,n})$ and $\varphi(z_{\delta,j})$ for $\nu, j \in \sigma$ and $n \in \omega$. Fix δ and let $\rho = f^*(\delta)$; without loss of generality $\rho \notin E$. Let F' (resp. F'_ρ) be the subgroup of F generated by $\{y_{\delta,\nu,n} : \nu \in E, n \in \omega\} \cup \{z_{\delta,j} : j < \sigma\} \cup \{x_\beta : \beta < \delta\}$ (resp. by $\{y_{\delta,\nu,n} : \nu \in E \cap \rho, n \in \omega\} \cup \{z_{\delta,j} : j < \rho\} \cup \{x_\beta : \beta < \delta\}$) and K' (resp., K'_ρ) the subgroup generated by $\{w_{\delta,\nu,n} : \nu \in E, n \in \omega\} \cup \{x_\beta : \beta < \delta\}$ (resp., by $\{w_{\delta,\nu,n} : \nu \in E \cap \rho, n \in \omega\} \cup \{x_\beta : \beta < \rho\}$). Now F'/K' is σ-free since E is non-reflecting, so $F'_\rho + K/K \cong F'_\rho/K'_\rho$ is free and hence K'_ρ is a summand of F'_ρ; then it is easy to extend $\psi \restriction \{w_{\delta,\nu,n} : \nu \in E \cap \rho, n \in \omega\} + \varphi \restriction \{x_\beta : \beta < \rho\}$ to $\varphi : F'_\rho \to \mathbb{Z}$. For $\nu \in E$ with $\nu > \rho$ we have $\varphi(x_{\eta_\delta(\nu+n)}) = \psi(w_{\delta,\nu,n})$ for all $n \in \omega$. For some m_ν, $\zeta_\nu(n) \geq \rho$ when $n \geq m_\nu$. Then we can satisfy the equations

$$\psi(w_{\delta,\nu,n}) = 2\varphi(y_{\delta,\nu,n+1}) - \varphi(y_{\delta,\nu,n}) - \varphi(z_{\delta,\zeta_\nu(n)}) + \varphi(x_{\eta_\delta(\nu+n)})$$

by setting $\varphi(y_{\delta,\nu,n}) = 0 = \varphi(z_{\delta,\zeta_\nu(n)})$ for $n \geq m_\nu$ and defining $\varphi(y_{\delta,\nu,n})$ by downward induction for $n < m_\nu$. ∎

Finally we can put the pieces together to prove:

Theorem 17. *There is a model of ZFC + GCH such that for every Whitehead group A of infinite rank, there is a Whitehead group H_A of cardinality $\leq |A|^+$ such that* $\mathrm{Ext}(H_A, A) \neq 0$.

Proof. By standard forcing methods (cf. [14], [6], [16], [17]) there is a model of ZFC + GCH such that the following holds (where S^λ_μ denotes the set of ordinals $< \lambda$ of cofinality μ):

(i) for every infinite successor cardinal $\lambda = \mu^+$ there is a stationary subset S of $S^\lambda_{\mathrm{cf}(\mu)}$ with a ladder system $\bar{\eta} = \{\eta_\delta : \delta \in S\}$ which satisfies ω-uniformization (or even κ-uniformization for every $\kappa < \mu$);

(ii) for every infinite successor cardinal $\lambda = \mu^+$ there is a stationary subset Y of $S^\lambda_{\mathrm{cf}(\mu)}$ such that $\diamondsuit_\lambda(Y)$ holds;

(iii) for every regular uncountable cardinal σ, there is a non-reflecting stationary subset E of S^σ_ω such that $\diamondsuit_\sigma(E')$ holds for every stationary subset E' of E;

(iv) there is a tree-like ladder system on a stationary subset of ω_1 which satisfies 2-uniformization but not ω-uniformization.

(In fact, we can get more: we can strengthen (i) and (ii) to the following: for every infinite successor cardinal $\lambda = \mu^+$ there is a normal ideal I_λ containing the non-stationary ideal such that for every $S \in I_\lambda$, $S - S^\lambda_{\mathrm{cf}(\mu)}$ is non-stationary, and there exists a stationary $S' \in I_\lambda$ disjoint from S; moreover, for every $S \in I_\lambda$, there is a ladder system $\bar{\eta} = \{\eta_\delta : \delta \in S\}$ which satisfies ω-uniformization and for every $S \notin I_\lambda$, $\diamondsuit_\lambda(S)$ holds.)

We work in this model. Let A be a Whitehead group of infinite rank. If the rank of A is $\aleph_0$, then A is isomorphic to $\mathbb{Z}^{(\omega)}$ and it is well-known (cf. [13], [5, XII.3.]) that (iv) implies that there is a Whitehead group H which is not $\aleph_1$-coseparable, i.e., $\mathrm{Ext}(H, \mathbb{Z}^{(\omega)}) \neq 0$. If the cardinality of A is either singular or a successor cardinal, then for $\lambda = |A|$ if $|A|$ is regular, or $\lambda = |A|^+$ if $|A|$ is singular, the properties (i), (ii) and (iii) allow us to build a group H_A of cardinality λ as in Definition 10, which is Whitehead by Theorem 16 and such that by Theorem 12, 14 or 15, $\mathrm{Ext}(H_A, A) \neq 0$.

It is also consistent to assume that there are no regular limit (i.e. inaccessible) cardinals, in which case we have covered all possibilities for the cardinality of A and we are done. Another approach is to allow inaccessible cardinals but force the model to satisfy in addition:

(v) for every inaccessible cardinal λ there is a stationary subset S of S^λ_ω with a ladder system $\bar{\eta} = \{\eta_\delta : \delta \in S\}$ which satisfies ω-uniformization; moreover $\diamondsuit_\lambda$ holds.

As in Lemma 8, one can show that $S(A)$ is stationary and then the proof is similar to that in Theorem 14. ∎

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