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Determined Theories and Limit Laws

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1 Introduction

Baldwin and Shelah [3] provided an alternative to Ehrehfreuht-games or the usual quantifier elimination arguments for proofs of completeness; they applied this method to prove the 0-1 law for first order logic edge probability $n^{-\alpha}$ and irrational α, $0 < \alpha < 1$. Baldwin [2] abstracted this argument into the definition of a *determined* theory. In this paper, we generalize the method and use it to give a new proof of convergence for the following edge probability p_n^l. Lubos Thoma and Joel Spencer in [4] studied the probability measures on graphs of size n given by edge probability:

$$p_n^l = \frac{ln(n)}{n} + \frac{l \cdot ln(ln(n))}{n} + \frac{c}{n}$$

where l ranges over all finite nonnegative integers, and c is a positive constant. Probabilities in this range govern the existence of isolated (finite degree) vertices in the random graph.

Recall that a family of edge measures p_n on graphs of size n obeys a 0-1 law (for first order logic) if for each first order sentence ϕ, $\lim_{n\to\infty} p_n(\phi)$ is either 0 or 1; the family p_n has a convergence law if each such limit converges.

Our interest in these graphs arose from the similarity between the models of the almost sure theories here and the theories considered in [1]. In particular, the limit models are rather simple from a model theoretic standpoint. They decompose into components which are 'almost' trees; the theories can be seen to be ω-stable. In this range of probabilities, the parameter l determines the possibility of the random graph admitting an 'r-isolated point': a vertex of degree r; there is none if $r < l$ and infinitely many if $r > l$. But for $r = l$, the number of vertices of degree r is not determined. In essence, fixing this number

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determines the completions of the almost sure theory. These, in turn, allow one to compute convergence for each sentence.

Lubos and Thoma proved:

Theorem 1.1 *1. A graph G satisfies the almost sure theory Th_l for p_n^l if:*

  (a) *The number of vertices in G is strictly less than the number of edges.*
  (b) *For all $t \geq 1$ and $m \geq 3$, t copies of a cycle on m vertices can be embedded in G.*
  (c) *For all integers r, s, $t \geq 0$, $s \geq r$, there do not exist a pair of vertices x, $y \in G$ such that x has degree r and y has degree s and the distance from x to y is equal to t.*
  (d) *For all integers r, $t \geq 0$ and $m \geq 3$ there do not exist a vertex $x \in G$ of degree r that is a distance t from a cycle on m vertices contained in G.*
  (e) *For all integers r, $0 \leq r < l$, there does not exist a vertex of degree $r \in G$.*
  (f) *For all integers $t \geq 1$ and $r > l$, there exist t vertices of degree $r \in G$.*

2. *Moreover, for any integer s, if the sentence $\sigma_{l,s}$ asserts "there exists precisely s vertices of degree l" then $\lim_{n\to\infty} p_n^l(\sigma_{l,s})$ exists.*

3. *For each l and s adding $\sigma_{l,s}$ to Th_l yields a complete theory.*

From this analysis of graphs with edge probability p_n^l they conclude a convergence law showing that the probability of any sentence is a finite sum of members of an easily described family.

Theorem 1.2 *Let $lim_{n\to\infty} p_n^l(\sigma_{l,s}) = q_{l,s}$. Then for any L-sentence θ, there exists a finite set I of nonnegative integers such that $lim_{n\to\infty} p_n^l(\theta) = \Sigma_{i\in I} q_{l,i}$ or $lim_{n\to\infty} p_n^l(\theta) = 1 - \Sigma_{i\in I} q_{l,i}$*

This paper is a step in isolating the 'model theoretic' from the 'probabilistic' components of proofs of limit laws on finite models. In Section 2 we give a general definition of an indexed closure operator and a determined theory. Relying on the probability arguments of Spencer and Thoma for parts one and two of Theorem 1.1, we give in Section 3 a different model theoretic proof of part three of Theorem 1.1 and of Theorem 1.2. In particular, the fact (and computation showing it) that $\Sigma_{i<\omega} q_{l,i} = 1$ which is a part of the Spencer-Thoma argument is avoided here.

We write $Mod(T)$ for the class of models of a theory T. The collection of finite subsets of a set X is denoted by $S_\omega(X)$. For any model M, and for any $\overline{a} \in M^r$, $\theta(M, \overline{a})$ denotes the set of solutions of $\theta(x, \overline{a})$ in M. We denote the length of a tuple $\overline{a}$ by $lg(\overline{a})$.

2 Indexed closure and determined theories

The key to the method of determined theories is a way of breaking the algebraic closure of a finite set into a (possibly infinite) sequence of finite sets: an indexed closure operator. We give here a general notion of such a closure operator and use it to provide a method to prove not only 0-1 but convergence laws. The closure operator of Definition 2.2 which is used in Section 3 to prove Theorem 3.15 and Theorem 3.17 is a special case.

If cl is a function from $\omega \times S_\omega(M) \rightarrow S_\omega(M)$, we write $\mathrm{cl}^i_M(\overline{a})$ for $cl(i, \overline{a})$.

Definition 2.1 An *indexed closure operator* Cl for a theory T is a function, which for each $M \models T$, maps $\omega \times S_\omega(M) \rightarrow S_\omega(M)$ and has the following properties.

1. For any model $M \in Mod(T)$ and $\overline{a} \subseteq \overline{b} \in M$, for $j < i < \omega$, $\mathrm{cl}^j_M(\overline{a}) \subseteq \mathrm{cl}^i_M(\overline{a})$ and $\mathrm{cl}^i_M(\overline{a}) \subseteq \mathrm{cl}^i_M(\overline{b})$.
2. For any $M, N \in Mod(T)$, if for some s, $\mathrm{cl}^s_M(\emptyset) \simeq \mathrm{cl}^s_N(\emptyset)$ then for all $0 \leq i \leq s$, $\mathrm{cl}^i_N(\emptyset) \simeq \mathrm{cl}^i_M(\emptyset)$.

We extend the notation by writing $\mathrm{cl}^\omega_M(\overline{a})$ for $\cup_{i<\omega}\mathrm{cl}^i_M(\overline{a})$. In the example considered in this paper, for all $M \in Mod(T)$, there exists a $k < \omega$, such that for all $s > k$, $\mathrm{cl}^k_M(\overline{a}) = \mathrm{cl}^s_M(\overline{a})$ so $\mathrm{cl}^\omega_M(\overline{a}) = \mathrm{cl}^k_M(\overline{a}) = \mathrm{acl}_M(\overline{a})$. Following is the indexed closure operator we will use in this paper. While it provides a natural way to 'layer' the algebraic closure, there is an unfortunate lack of monotonicity, described in Example 3.7, which require us to treat closure over the empty set with special care.

Definition 2.2 For $k \leq \omega$, let the k-closure of $\overline{a}$ in M, $cl^k_M(\overline{a})$, be the set of solutions in M of all the formulas $\theta(x, \overline{a})$ where the quantifier rank of θ and $|\theta(M, \overline{a})|$ are each less than k.

The following fact, shown by straightforward calculation [3], is fundamental for the kind of argument used here.

Lemma 2.3 *For any first order T in a finite relational language. there exists a function f of $|A|, m, n$, such that for any $M \models Th_l$ and any embedding of A into M, $cl^m_M(\mathrm{cl}^n(A)) \subseteq cl^{f(|A|,m,n)}_M(A)$.*

In the following definition, we vary from the [3] by treating the closure of the empty set in a special way. The necessity of this special treatment is shown in Example reffunnycondition

Definition 2.4 1. We write $\simeq$ for isomorphism. We say $\mathrm{cl}^k_M(\overline{a}) \simeq^s \mathrm{cl}^k_{M'}(\overline{a}')$ if $\mathrm{cl}^k_M(\overline{a}) \simeq \mathrm{cl}^k_{M'}(\overline{a}')$ by an isomorphism taking $\overline{a}$ to $\overline{a}'$ and $\mathrm{cl}^k_M(\emptyset) \simeq \mathrm{cl}^k_{M'}(\emptyset)$

2. For an integer k and a theory T, a formula $\theta(\overline{x})$ is *determined* by its k-closure in T if for any $M, M' \models T$ and for any $\overline{a} \in M^r$ and $\overline{a}' \in M'^r$, if $\mathrm{cl}^k_M(\overline{a}) \simeq^s \mathrm{cl}^k_{M'}(\overline{a}')$, then $M \models \theta(\overline{a})$ if and only if $M' \models \theta(\overline{a}')$.
3. The theory T is *determined* if for any formula $\theta(\overline{x})$, there is an integer k_θ such that $\theta(\overline{x})$ is determined by its k_θ-closure in T.

3 An Application

In this section we consider the almost sure theory studied by Spencer and Thoma in [4], whose axioms are the theory Th_l of the introduction. Let $Th_{l,s}$ denote the theory which consists of Th_l plus the axiom "there exists s isolated vertices of degree l' and $Th_l^{>0}$ denote the theory which consists of Th_l plus the axiom "there exists an isolated vertex of degree l".

Notation 3.1 A tree in which every vertex has infinite degree is denoted by T and called a *complete tree*. A *hairy cycle* is a cycle with a complete tree attached to every vertex of the cycle. Let H_n denote a hairy cycle whose cycle is of size n. An isolated component, denoted I_n is a tree which contains one point with degree n and all others have infinite degree.

As was pointed out in [4] it is easy to check that each model of Th_l is a direct sum of the following components:

1. For every integer i greater than one, infinitely many components each containing one cycle of size i and every vertex has infinite degree.
2. For every $r > l$, infinitely many components which do not contain a cycle and every vertex has infinite degree except one, which has degree r.
3. Any (possibly finite) number of components which do not contain a cycle and every vertex has infinite degree.
4. For some $s > 0$, s copies of components which do not contain a cycle and every vertex has infinite degree except one vertex which has degree l.

More formally:

Lemma 3.2 *Let M be a countable model of Th_l, then there exists an s, $0 \leq s < \omega$, and j, $0 \leq j \leq \omega$, such that M has the following form:*

$$\Sigma_{1<i<\omega} H_i^{(\omega)} \oplus \Sigma_{l<i<\omega} I_i^{(\omega)} \oplus T^{(j)} \oplus I_l^{(s)}$$

For $0 \leq s \leq \omega$, we denote the model with the form above and s copies of I_l by M_s.

Remark 3.3 From Lemma 3.2 we observe the following facts about algebraic closure. For any model M of $Th_{l,s}$ with $s < \omega$:

1. $\mathrm{acl}_M(\emptyset)$ consists of the isolated vertices of degree l, and their neighbors.
2. $\mathrm{acl}_M(a) = \mathrm{acl}_M(\emptyset)$ unless a is on component which contains a cycle or an isolated point.
    (a) In the first case $\mathrm{acl}_M(a)$ is the union of $\mathrm{acl}_M(\emptyset)$ with all points on the path from a to the cycle.
    (b) In the second case $\mathrm{acl}_M(a)$ is the union of $\mathrm{acl}_M(\emptyset)$ with all points on the path from a to the isolated point.
3. $\mathrm{acl}_M(a, b) = \mathrm{acl}_M(a) \cup \mathrm{acl}_M(b)$ unless a and b are on the same component; in that case it also includes all points on the path from a to b.
4. For any set A, $\mathrm{acl}_M(A) = \bigcup_{a,b \in A} \mathrm{acl}_M(a, b)$.

Definition 3.4 Let G^l_n, for $n > 0$, be the direct sum of n components, each consisting of one vertex with l neighbors. To ease notation, define G^l_ω to be equal to the empty set.

It is easy to see that for any $M \models Th^{>0}_l$, there exists an s, $0 < s \leq \omega$ such that $\mathrm{cl}^\omega_M(\emptyset) \simeq G^l_s \simeq \mathrm{cl}^t_M(\emptyset)$ for $t \geq s$. The following property of k-closure in models of Th_l is crucial. Let M be a model of Th_l, and let $n = |acl_M(\emptyset)|$; if $k < n$, $\mathrm{cl}^k_M(\emptyset) \simeq \emptyset$. So, if M_0 models $Th_{l,0}$, $\mathrm{cl}^k_M(\emptyset) \simeq \emptyset \simeq \mathrm{cl}^k_{M_0}(\emptyset)$. In particular, for any $k < \omega$ we have $\mathrm{cl}^k_{M_\omega}(\emptyset) \simeq \emptyset \simeq \mathrm{cl}^k_{M_0}(\emptyset)$ and $M_\omega \not\equiv M_0$. Thus the theory Th_l is not determined. However, we will show that $Th_{l,0}$ and $Th^{>0}_l$ are each determined with respect to our notion of closure.

Notation 3.5 We adopt the following notation. For any a in M, let $C_M(\overline{a})$ be the union of the components in M which intersect $\overline{a}$. Denote the number of free variables plus the quantifier rank of a formula θ by $qr^*(\theta)$.

Definition 3.6 If $\theta(\overline{y})$ is quantifier free, $k_\theta = (l+1) \cdot \mathrm{lg}(\overline{y})$. If $\theta(\overline{y})$ is the formula $(\exists x)\phi(x, \overline{y})$ let k_θ be the least integer $k > \max(3k_\phi, (l+1)qr^*(\phi))$ and such that for for any element b in $\mathrm{cl}^{k_\phi}(\overline{a})$, $\mathrm{cl}^{k_\phi}(b, \overline{a}) \subseteq \mathrm{cl}^{k_\theta}(\overline{a})$.

Our main induction concerns a formula $\phi(x, \overline{y})$; we denote $\exists x \phi(x, \overline{y})$ by $\theta(\overline{y})$.

The condition $k_\theta > (l+1)qr^*(\phi)$ guarantees that if $cl^{k_\theta}_M(\emptyset) \simeq cl^{k_\theta}_{M'}(\emptyset)$ and they each contain an isolated point, then they contain the same number of isolated points.

The following example shows why we had to treat the closure of the empty set in a special way in Definition 2.4.

Example 3.7 Consider the models M_1 and M_2 with the notation set after Lemma 3.2. Let a and a' be neighbors of neighbors of isolated vertices in M_1 and M_2 respectively. Then the $l+2$ closure of a and the $l+2$ closure of a' are isomorphic. Both consist of the neighbors of the isolated point near a, (a') respectively. But the $l+2$ closure of the empty set is empty in M_2 and contains the neighbors of the isolated point in M_1.

If we tried to define determined only by looking at isomorphic closures rather than insisting on isomorphism between the closure of the empty set as well, $Th_l^{>0}$ would not be determined. To see this consider the formula $\phi(x,y)$ which asserts each of x and y are distance 2 from an isolated point but the distance between them is at least 5. Then, estimating quite roughly, $k_\theta < 15l$. Let n be much greater than $15l$ and let M be the model M_n and M' the model M_1. Let $a \in M$ and $a' \in M'$ both be distance 2 from an isolated point. Then $\mathrm{cl}^{k_\theta}_M(a) \simeq \mathrm{cl}^{k_\theta}_M(a')$. But $M \models \theta(a)$ and $M' \not\models \theta(a')$.

The main result is to show in Theorem 3.9 that the theories $Th_l^{>0}$ and $Th_{l,0}$ are determined. This argument is simply a different way to organize the back-and-forth argument showing each $Th_{l,s}$ is complete. We require one technical definition.

Definition 3.8 *For any M and $\overline{a}, b \in M$, let $d(b, \mathrm{cl}^{k_\phi}_M(\overline{a}))$ be the shortest distance from b to $\mathrm{cl}^k_M(\overline{a})$. For any M and $\overline{a} \in M$, let*

$$D^{\phi}_{M,\overline{a}} = \max\{d(b, \mathrm{cl}^{k_\theta}_M(\overline{a})) : b \in \phi(M, \overline{a})\}$$

.

Theorem 3.9 *Suppose both M and M' are models of $Th_l^{>0}$ or both are models of $Th_{l,0}$. For any $\theta(\overline{x})$ in L (with arity r), and any $\overline{a}$, $\overline{a}$ in M^r and M'^r respectively, there exists k_θ such that if $\mathrm{cl}^{k_\theta}_M(\overline{a}) \simeq^s cl^{k_\theta}_{M'}(\overline{a}')$ then $M \models \theta(\overline{a})$ if and only if $M' \models \theta(\overline{a}')$.*

Proof: The lemma follows by induction on the complexity of formulas. Let $\theta(\overline{y}) = \exists x \phi(x; \overline{y})$. Choose k_θ as in Definition 3.6. Let $\overline{a}$, $\overline{a}'$ be in M and M' respectively. We need to show, if $\mathrm{cl}^{k_\theta}_M(\overline{a}) \simeq^s cl^{k_\theta}_{M'}(\overline{a}')$ and $\theta(\overline{a})$ holds in M, then we can choose b, satisfying $\phi(x, \overline{a})$, such that there exists b' for which $\mathrm{cl}^{k_\phi}_M(\overline{a}, b) \simeq^s cl^{k_\phi}_{M'}(\overline{a}', b')$ (equivalently, since $\mathrm{cl}^{k_\theta}_M(\overline{a}) \simeq^s cl^{k_\theta}_{M'}(\overline{a}')$, $\mathrm{cl}^{k_\phi}_M(\overline{a}, b) \simeq cl^{k_\phi}_{M'}(\overline{a}', b')$) whence by induction $M \models \phi(b, \overline{a})$ if and only if $M' \models \phi(b', \overline{a}')$.

Now, all possible cases are handled by the next 4 lemmas. The major division depends on whether $D^{\phi}_{M,\overline{a}} > k_\phi$. Within each side of this dichotomy, there are several cases depending on the disjoint cases: $\mathrm{cl}^{k_\phi}(b)$ is $\{b\}$, or contains a cycle, or contains an isolated point.

First we consider the case where $D^{\phi}_{M,\overline{a}}$ is large and $\mathrm{cl}^{k_\phi}(b)$ is either $\{b\}$ or contains a cycle.

Lemma 3.10 *Let $M, M' \models Th_l$ and $\overline{a} \in M^r$. Suppose $D^{\phi}_{M,\overline{a}} > k_\phi$. Fix $b \in M$ for which $d(b, \mathrm{cl}^{k_\theta}_M(\overline{a})) = D^{\phi}_{M,\overline{a}}$. Suppose $\mathrm{cl}^{k_\phi}_M(b) = \{b\}$ or $\mathrm{cl}^{k_\phi}_M(b)$ contains a cycle. If $\mathrm{cl}^{k_\theta}_M(\overline{a}) \simeq^s \mathrm{cl}^{k_\theta}_{M'}(\overline{a}')$, there exists a $b' \in M'$ such that $\mathrm{cl}^{k_\phi}_M(\overline{a}, b) \simeq^s \mathrm{cl}^{k_\phi}_{M'}(\overline{a}', b')$.*

Proof: By assumption, $\mathrm{cl}^{k_\phi}_M(b)$ is either just b or the vertex b and a cycle of cardinality less than k_ϕ and the vertices on a path, of length less than k_ϕ, from b to this cycle. In the first case, choose b' to be a vertex of infinite degree, on a component which does not intersect $\overline{a}'$. In the second case, choose one of the infinitely many components in M' which contains an n-cycle and does not intersect $\overline{a}'$, and choose b' on this component with the same distance from the n-cycle as b is from the n-cycle on the component where b resides. In both cases, the result follows since $\mathrm{cl}^{k_\phi}(\overline{a}, b) = \mathrm{cl}^{k_\phi}(\overline{a}) \cup \mathrm{cl}^{k_\phi}(b)$ and similarly for $\overline{a}', b'$. $\square$

Now we consider the case where $D^{\phi}_{M,\overline{a}}$ is large and $\mathrm{cl}^{k_\phi}(b)$ contains an isolated point. There are two traps which must be avoided in the following proof: $M = M_\omega$ and $M' = M_0$, $M = M_i$ and $M' = M_j$ where j is much greater than k_θ is much greater than i. We avoid the first by restricting to $Th^{>0}_l$; this is permissible since there are no isolated points in models of $Th_{l,0}$ and so the case can occur only for $Th^{>0}_l$. The second is dealt with by using $\simeq^s$.

Lemma 3.11 *Let $M, M' \models Th^{>0}_l$ and $\overline{a} \in M^r$. Suppose $D^{\phi}_{M,\overline{a}} > k_\phi$. Fix $b \in M$ for which $d(b, \mathrm{cl}^{k_\theta}_M(\overline{a})) = D^{\phi}_{M,\overline{a}}$. Suppose there is an isolated vertex contained in $\mathrm{cl}^{k_\phi}_M(b)$. If $\mathrm{cl}^{k_\theta}_M(\overline{a}) \simeq^s \mathrm{cl}^{k_\theta}_{M'}(\overline{a}')$, there exists a $b' \in M'$ such that $\mathrm{cl}^{k_\phi}_M(\overline{a}, b) \simeq^s \mathrm{cl}^{k_\phi}_{M'}(\overline{a}', b')$.*

Proof. Since $D^{\phi}_{M,\overline{a}} > k_\phi$, $\mathrm{cl}^{k_\phi}_M(\overline{a}, b) = \mathrm{cl}^{k_\phi}_M(\overline{a}, b) \cup c^{k_\phi}_M(b)$. If there is an isolated point c' in $M' - \mathrm{cl}^{k_\theta}_{M'}(\overline{a}')$, this is easy, we can map b to a point near c'. Specifically, since $k_\theta > l + 1$, none of the neighbors of c' can be in $\mathrm{cl}^{\theta}_{M'}(\overline{a}')$ either. Therefore, there exists a b' such that $\mathrm{cl}^{k_\phi}_{M'}(\overline{a}', b') = \mathrm{cl}^{k_\phi}_M(\overline{a}', b') \cup \mathrm{cl}^{k_\phi}_M(b')$ and $\mathrm{cl}^{k_\phi}_{M'}(b') \simeq \mathrm{cl}^{k_\phi}_M(b)$. So $\mathrm{cl}^{k_\phi}_{M'}(\overline{a}', b') \simeq \mathrm{cl}^{k_\phi}_{M'}(\overline{a}', b')$.

We are left with the case that all isolated points of M' are in $\mathrm{cl}^{k_\theta}_{M'}(\overline{a}')$. But then all isolated points of M' are in $\mathrm{cl}^{k_\theta}_{M'}(\emptyset)$. Since $\mathrm{cl}^{k_\theta}_M(\overline{a}) \simeq^s \mathrm{cl}^{k_\theta}_{M'}(\overline{a}')$, $\mathrm{cl}^{k_\theta}_M(\emptyset) \simeq \mathrm{cl}^{k_\theta}_{M'}(\emptyset)$: this is the essential use of $\simeq^s$ instead of $\simeq$. Since $\mathrm{cl}^{k_\theta}_{M'}(\emptyset)$ contains all the $t > 0$ isolated points in M', $\mathrm{cl}^{k_\theta}_M(\emptyset)$ and therefore $\mathrm{cl}^{k_\theta}_M(\overline{a})$ contains all the $t > 0$ isolated points in M. So $b \in C_M(\overline{a})$. Let $\overline{a}_0 = C_M(b) \cap \overline{a}$. Since $d(b, \mathrm{cl}^{k_\theta}_M(\overline{a})) = D^{\phi}_{M,\overline{a}} > k_\phi$, the isolated point c of $C_M(b)$ is not in $\mathrm{cl}^{k_\theta}_M(\overline{a}_0)$. So we can map b to a b' on the component of $\overline{a}'_0$ so that $\mathrm{cl}^{k_\phi}_{M'}(\overline{a}', b') \simeq \mathrm{cl}^{k_\phi}_{M'}(\overline{a}', b')$.

Now we turn to the cases where $D^{\phi}_{M,\overline{a}}$ is small; first, suppose $\mathrm{cl}^{k_\phi}(b) = \{b\}$.

Lemma 3.12 *Let $M, M' \models Th_l$ and $\overline{a} \in M^r$. Suppose $D^{\phi}_{M,\overline{a}} \leq k_{\phi}$. Fix $b \in M$ for which $d(b, \mathrm{cl}^{k_\theta}_M(\overline{a})) = D^{\phi}_{M,\overline{a}}$. For any $\overline{a}' \in M'^r$, $b \in M$ if $\mathrm{cl}^{k_\phi}_M(b) = \{b\}$ and $\mathrm{cl}^{k_\theta}_M(\overline{a}) \simeq^s \mathrm{cl}^{k_\theta}_{M'}(\overline{a}')$, there exists a $b' \in M'$ such that $\mathrm{cl}^{k_\phi}_M(\overline{a}, b) \simeq^s \mathrm{cl}^{k_\phi}_{M'}(\overline{a}', b')$.*

Proof: We may assume that $b \notin \mathrm{cl}^{k_\theta}_M(\overline{a})$, otherwise the result follows immediately from the second requirement in defining k_θ. First we claim there is at most one path whose vertices are in $\mathrm{cl}^{k_\phi}_M(b, \overline{a}) - \mathrm{cl}^{k_\theta}_M(\overline{a})$ from b to the k_θ-closure of $\overline{a}$ in M. Suppose not, then we claim that any vertex c which lies on a fork of the path from b to the $\mathrm{cl}^{k_\theta}_M(\overline{a})$ is in fact in $\mathrm{cl}^{k_\theta}_M(\overline{a})$, which is a contradiction.

So assume there are two paths in $\mathrm{cl}^{k_\phi}_M(b, \overline{a})$ from c to $\mathrm{cl}^{k_\theta}_M(\overline{a})$, one going to say, a_1 in $\overline{a}$, and the other going to say, a_2 in $\overline{a}$, (a_1 could be equal to a_2) with lengths k_1 and k_2 respectively. Note, since both paths are in $\mathrm{cl}^{k_\phi}(b, \overline{a})$, then we may assume both k_1 and k_2 are less than k_ϕ which is less than k_θ. Thus there is at most one other vertex with distance k_1 to a_1 and distance k_2 to a_2 (if $a_1 = a_2$, c is the only vertex, since a component of a model of Th_l can have at most one cycle). Thus c satisfies a formula with only two solutions (or one solution if $a_1 = a_2$) and quantifier rank less than k_ϕ. So $c \in \mathrm{cl}^{k_\theta}_M(\overline{a})$. This proves the claim.

Assume the shortest path from b to $\mathrm{cl}^{k_\phi}_M(\overline{a})$ is of length k_0 and the nearest vertex in $\mathrm{cl}^{k_\phi}_M(\overline{a})$ to b is a_0. Let a_0' be in $\mathrm{cl}^{k_\phi}_{M'}(\overline{a}')$ such that a_0' corresponds to a_0 in the isomorphism from $\mathrm{cl}^{k_\theta}_M(\overline{a})$ to $\mathrm{cl}^{k_\theta}_{M'}(\overline{a}')$.

We need to choose a vertex $b' \in M'$ a vertex and a path in M' of size k_0 none of whose vertices are in $\mathrm{cl}^{k_\phi}_{M'}(\overline{a}')$ except for a_0'. This is immediate if a_0' has infinite degree; a_0 and a_0' have the same degree. If a_0 has finite degree then all the neighbors of a_0 are in $\mathrm{cl}^{k_\phi}_M(\overline{a})$ since $k_\phi > l$. But then k_0 was not chosen minimal. So a_0 and thus a_0' has infinite degree and we can choose b'.

Finally by the first claim, $\mathrm{cl}^{k_\phi}_M(b, \overline{a})$ is contained in $\mathrm{cl}^{k_\phi}_M(\overline{a})$ along with b and a path of length at most k from b to $\mathrm{cl}^{k_\theta}_M(\overline{a})$ and $\mathrm{cl}^{k_\phi}_{M'}(b', \overline{a}')$is contained in $\mathrm{cl}^{k_\theta}_{M'}(\overline{a}')$ along with b' and a path from b' to $\mathrm{cl}^{k_\theta}_{M'}(\overline{a}')$ of the same length. Therefore we have $\mathrm{cl}^{k_\phi}_M(b, \overline{a}) \simeq^s \mathrm{cl}^{k_\phi}_{M'}(b', \overline{a}')$ □

Finally, we consider the case where $D^{\phi}_{M,\overline{a}}$ is small and $\mathrm{cl}^{k_\phi}(b) = \{b\}$ contains a cycle or an isolated point.

Lemma 3.13 *Let $M, M' \models Th_l$. Fix $\overline{a} \in M^r$, $\overline{a}' \in M'^r$, $b \in M$ with $D^{\phi}_{M,\overline{a}} \leq k_\phi$. If there is an isolated vertex or a cycle contained in $\mathrm{cl}^{k_\phi}_M(b)$ then the isolated vertex or the cycle is contained in $\mathrm{cl}^{k_\theta}_M(\overline{a})$. In particular, if $\mathrm{cl}^{k_\theta}_M(\overline{a}) \simeq^s \mathrm{cl}^{k_\theta}_{M'}(\overline{a}')$, there exists a $b' \in M'$ such that $\mathrm{cl}^{k_\phi}_M(\overline{a}, b) \simeq^s \mathrm{cl}^{k_\phi}_{M'}(\overline{a}', b')$.*

Proof: Note that $\overline{a}$ satisfies the formula which asserts: there is a path of length at most k_ϕ to a point x and there is a path of length at most k_ϕ from x

to a cycle of length at most k_ϕ (or to an l-isolated point). Since $k_\theta > 3k_\phi$ and $l < k_\phi$ the result follows.

This completes the proof of Theorem 3.9. We now apply this result to computing the probabilities of sentences with respect to p^l_n.

Definition 3.14 *Let σ^l_s be the sentence: "there exists exactly s vertices of degree l". Define $q_{l,s}$ to be the limit probability of σ^l_s (the existence of the limit is shown in [4].) Note this limit also depends on the constant c in the definition of p^l_n.*

Theorem 3.15 *For every non-negative integer s and for $s = \omega$, $Th_{l,s}$ is a complete theory. Furthermore these are all possible completions of the almost sure theory Th_l.*

Proof: First we show $\{Thl, s : 0 < s \leq \omega\}$, is the set of all possible completions of the theory $Th_l^{>0}$. Fix an integer s or let $s = \omega$. It is clear that if M and M' model $Th_{l,s}$, then $\mathrm{cl}^\omega_M(\emptyset) \simeq \mathrm{cl}^\omega_{M'}(\emptyset) \simeq G^l_s$. (Remember, by convention, $G^l_\omega = \emptyset$.) Furthermore, for all $t \geq s$, $\mathrm{cl}^t_M(\emptyset) \simeq \mathrm{cl}^t_{M'}(\emptyset) \simeq G^l_s$, since t is large enough to capture the algebraic closure of M and M'. Finally, for all $q < s$ $\mathrm{cl}^q_M(\emptyset) \simeq \mathrm{cl}^q_{M'}(\emptyset) \simeq \emptyset$. Thus, since for all t, $0 < t \leq \omega$, $\mathrm{cl}^t_M(\emptyset) \simeq \mathrm{cl}^t_{M'}(\emptyset)$, Theorem 3.9 implies $M \equiv M'$. Therefore $Th_{l,s}$ is complete. Since $Th_l^{>0}$ is determined, $\{Th_{l,s} : 0 < s \leq \omega\}$ is the set of all completions of $Th_l^{>0}$.

We note now that $Th_{l,0}$ is the theory Th_l plus the negation of the axiom "there exists an isolated vertex"(recall, this axiom plus Th_l is $Th_l^{>0}$). Since $Th_{l,0}$ is determined, and the algebraic closure of the empty set of any model of $Th_{l,0}$ is empty, $Th_{l,0}$ is complete. Thus we have now all possible completions of Th_l. □

The existence of k^* below follows from our characterisation of the closure of the empty set in models of Th_l.

Definition 3.16 For any k, let k^* be the least integer s greater than or equal to k such that for any $t \geq s$, $\mathrm{cl}^k_{M_t}(\emptyset) = \emptyset$.

We write $p^l(\theta)$ for $lim_{n\to\infty} p^l_n(\theta)$ if it exists.

Theorem 3.17 *For any L-sentence θ, there exists a finite set I of positive integers such that $p^l(\theta)$ is one of $\Sigma_{i\in I} q_{l,i}$, $p^l(\theta) = 1 - \Sigma_{i\in I} q_{l,i}$, or $q^l_0 + 1 - \Sigma_{i\in I} q_{l,i}$.*

Proof: Note that $p^l(\theta) = p_l(\sigma^l_0 \to \theta) \times p_l(\sigma^l_0) + p_l(\neg\sigma^l_0 \to \theta) \times p_l(\neg\sigma^l_0)$ if this limit exists. The first term always exists and is either 0 or $p_l(\sigma^l_0)$ since $Th_{l,0}$ is complete.

We compute the second term using that $Th_l^{>0}$ is determined. By Theorem 3.9, there exists a finite k_θ such that the k_θ-closure of the empty set determines θ in $Th_l^{>0}$. Without loss of generality, assume that $M_{k^*_\theta} \models \theta$ (an analogous argument works if $M_{k^*_\theta} \models \neg\theta$). Consider the set I (possibly empty) of nonnegative integers bounded by k^*_θ such that for all $i \in I$, $Th_{l,i}\ models \neg\theta$. If this set is empty we conclude by Lemma 2.2 and Theorem 3.15, that $Th_l^{>0}$ proves θ. In this case, it is clear that $lim_{n\to\infty} p^l(\theta) = 1 - q_0^l$ or 1, depending on whether $Th_{l,0} \models \theta$. If however the set I is not empty $Th_l^{>0}$ proves $\vee_{i\in I}\sigma_i^l \leftrightarrow \neg\theta$. Thus $p^l(\theta)$ is $1 - \Sigma_{i\in I} q_{l,i}$ plus the contribution from $Th_{l,0}$. $\square$

4 Conclusion and Questions

We have provided another proof of the convergence law for the edge probability considered by [4]. Our analysis allows for one less probability calculation. But the argument depends very heavily on Lemma 3.16 which is seems an unusual and overly strong condition. In particular, it implies that in every model the algebraic closure of the empty set is finite. This seems to be a necessary condition for this type of argument to work. Basically, the key is to be able to compute the probability of assertions, '$\mathrm{acl}_M(\emptyset)$ has form X'. Can a general method of showing convergence be developed by adding this hypothesis?

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