# ON A CLASS OF ALMOST DISJOINT PERMUTATION FAMILIES

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**ABSTRACT.** We construct several forcing models in each of which there exists a maximal cofinitary group, i.e., a maximal almost disjoint group in $G \leq Sym(\mathbb{N})$, such that $G$ is also a maximal almost disjoint family in $Sym(\mathbb{N})$. We shall also ask several open questions in this area in the fourth section of this paper.

## 1. Introduction.

We consider two kinds of closely related mathematical structures in this paper: cofinitary groups and almost disjoint (a.d.) permutation families in $Sym(\mathbb{N})$. The paper is a continuation of author's previous papers [Z], [Z1] and [Z2].

We first define a.d. permutation families in $Sym(\mathbb{N})$. We say that two permutations $f, g \in Sym(\mathbb{N})$ are a.d. with each other if and only if $|f \cap g| < \omega$, that is, that

$$\{n \in \mathbb{N} \mid f(n) = g(n)\}$$

is finite. And an a.d. permutation family $A \subseteq Sym(\mathbb{N})$ is a subset of $Sym(\mathbb{N})$ such that $f, g$ are a.d. for any $f, g \in A$.

Now let me give a brief introduction to cofinitary permutation groups. A permutation $g \in Sym(\mathbb{N})$ is cofinitary if and only if $g$ has only finitely many fixed points. A group $G \leq Sym(\mathbb{N})$ is cofinitary if and only if every non-identity element is cofinitary. It is easily seen that $G \leq Sym(\mathbb{N})$ is cofinitary if and only if $G$ is both an almost disjoint set of permutations and a group. For a discussion of different aspects of cofinitary groups, the reader can consult the well-written survey paper by P. Cameron (see [C]). Since the union of a chain of cofinitary permutation groups is cofinitary, Zorn's Lemma implies that maximal cofinitary groups exist, and indeed any cofinitary group is in a maximal one. Let me state a theorem which was proved by Adeleke (see [A]) and Truss (see [T] and [T1]).

**Theorem 1.1.** *If $G \leq Sym(\mathbb{N})$ is a maximal cofinitary group, then $G$ is not countable.*

Also, P. Neumann showed the following result.

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**Theorem 1.2.** *There exists a cofinitary group of cardinality* $2^{\aleph_0}$.

*Proof.* For details, see e.g. Proposition 10.4 in [C].

□

This motivated P. Cameron [C] to ask

**Question 1.3.** *If the Continuum Hypothesis (CH) fails, does there exist a maximal cofinitary group* $G$ *such that* $|G| < 2^{\aleph_0}$ *?*

I showed this question is independent of ZFC in [Z] (or [Z1]). Later on, in [BSZ], we proved following stronger results.

**Theorem 1.4.** *Every maximal cofinitary groups has size at least the cardinality of the smallest non-meager set of reals.*

**Note.** From the well–known fact that Martin's Axiom (MA) implies every non–meager set of reals has size $2^{\aleph_0}$, we also know the following

**Corollary 1.5.(MA).** *Every maximal cofinitary groups has size* $2^{\aleph_0}$.

**Theorem 1.6.** *Let* $M \models (ZFC{+}GCH)$. *In* $M$, *let* $C$ *be a closed set of uncountable cardinals with* $\aleph_1 \in C$, $\kappa \in C$ *for* $\aleph_1 \leq k \leq |C|$, *and* $\lambda^+ \in C$ *for* $\lambda \in C$ *with* $cf(\lambda) = \omega$. *Then there is a c.c.c. p.o. set* $\mathbb{P}$ *forcing* $max(C) = 2^{\aleph_0}$ *and* $Spec(mcg) = C$, *where* $Spec(mcg)$, *the spectrum of cardinalities of maximal cofinitary groups, denotes the set of cardinals* $\lambda$ *such that there is a maximal cofinitary group of size* $\lambda$.

Notice that we can also prove the corresponding results of 1.4, 1.5 and 1.6 for a.d. permutation families (for details, see [Z] and [BSZ]).

In [Z2], I tried to show that these two closely related structures are somehow different. By a rather complicated forcing argument, I proved the following result.

**Theorem 1.7.** *Let* $M \models (ZFC + \neg CH + MA)$. *Let* $\kappa, \lambda \in M$ *be cardinals such that* $\aleph_1 \leq \kappa < 2^{\aleph_0} = \lambda$. *Then there exists a c.c.c. notion of forcing* $\mathbb{P}$ *such that the following holds in* $M^{\mathbb{P}}$.

1. $2^{\aleph_0} = \lambda$,
2. *there exists a maximal cofinitary group* $G \leq Sym(\mathbb{N})$ *of cardinality* $\kappa$,
3. *thre exists an a.d. family* $A \subseteq Sym(\mathbb{N})$ *such that* $G \subseteq A$ *and* $|G| < |A|$.

Simon Thomas also observed the following result.

**Theorem 1.8.(MA).** *There exist a maximal cofinitary group* $G \leq Sym(\mathbb{N})$ *and a maximal almost disjoint permutation family* $A \subseteq Sym(\mathbb{N})$ *such that* $G$ *is a proper subset of* $A$.

However, it is quit often that the following questions were asked to the author during the talks of various conferences, mathematical seminars and colloquiums, ... etc..

**Quesion 1.9.** *Does there exist any maximal cofinitary group $G \leq Sym(\mathbb{N})$ such that $G$ is also a maximal almost disjoint permutation family in $Sym(\mathbb{N})$?*

In this paper, I shall prove that in several forcing models, Quesion 1.9 can be answered positively. In the last section, I shall also post several questions which was asked by variouse people and the author consider them hard to answer. The set–theoretical notation we use in this paper will follow from [Kun] or [Jech]. Thus if $\mathbb{P}$ is a notion of forcing and $p, q \in \mathbb{P}$, then $q \leq p$ means that $q$ is a strengthening of $p$. $M$ always denotes a countable models of $ZFC$. We sometimes denote the generic extension by $M^{\mathbb{P}}$ if we do not wish to specify a particular generic set $G$.

## 2. The forcing p.o. set which adjoins cofinitary permutations.

In this section, we introduce the forcing p.o.set which we shall use throughout the rest of the paper and we shall prove some technical result about the p.o.set.

**Definition 2.1.** Let $G \leq Sym(\omega)$ be a cofinitary group. Then the partially ordered set $\mathbb{G}_G$ consists of all conditions of the form $\langle s, F \rangle$ such that

(1) $s$ is a 1–1 finite partial function from $\omega$ to $\omega$;
(2) $F$ is a finite subset of $W_G$.

Here $W_G$ is a set consists of the words

$$w(x) = g_1 x^{n_1} ... g_t x^{n_t} g_{t+1}$$

which actually involve $x$ such that $g_l \in G \setminus \{id\}$ except possibly $g_1 = id$ or $g_{t+1} = id$, and $n_i \in \mathbb{Z} \setminus \{0\}$. We say $w_0$ is a conjugate subword of $w$ if

$$w(x) = u w_0 u^{-1}.$$

where $w_0 \in W_G \cup (G \setminus \{id\})$. We define $\langle s_2, F_2 \rangle \leq \langle s_1, F_1 \rangle$ if and only if

(a). $s_1 \subseteq s_2$ and $F_1 \subseteq F_2$;

(b). For every conjugate subword $w_0$ of $w \in F$, the following holds. Suppose that $w_0(s_1)(l) \uparrow$ and $w_0(s_2)(l) = (l)$. Then $w_0 = uzu^{-1}$ and $\langle l, n \rangle \in u^{-1}(s_2)$ for some $n$ with $z(s_1)(n) = n$. Here we use $\uparrow$ to denote that the computation is undefined.

**Note.** More explanation about the p.o.set $\mathbb{G}_G$ can be found in [Z1]. For example, by a complicated density argument, we proved the following result.

**Theorem 2.2.** *Let $G \leq (Sym(\mathbb{N}))^M$ be any cofinitary permutation group. Then there exists a generic permutation $g \in Sym(\mathbb{N}) \setminus G$ such that $\langle G, g \rangle = G * \langle g \rangle$ is a cofinitary permutation group. Here $G * \langle g \rangle$ denotes the free product of $G$ and $g$.*

Next, we shall prove some other technical lemmas concerning this p.o.set.

**Lemma 2.3.** *Let $G$ be a cofinitary group and let $f \in Sym(\mathbb{N}) \setminus G$ be such that $G \cup \{f\}$ is an almost disjoint family of $Sym(\mathbb{N})$. Assume that $w(x) \in W_G$ and $F = \{w\}$. If $\langle s, F \rangle \in \mathbb{G}_G$, then for all but finitely many $\langle n, f(n) \rangle \in f$,*

$$\langle s \cup \{\langle n, f(n) \rangle\}, F \rangle \leq \langle s, F \rangle.$$

*Proof.* Since $G \cup \{f\}$ is an almost disjoint family in $Sym(\mathbb{N})$ and $f \in Sym(\mathbb{N}) \setminus G$. We know that $f$ is a cofinitary permutation, i.e.,

$$| f \cap id | < \omega.$$

Let

$$w_0 = g_1 x^{n_1} g_2 x^{n_2} ... g_t x^{n_t} g_{t+1}$$

be a conjugate subword of $w$, where $n_i \in \mathbb{Z}\{0\}$ and $g_i \neq id$ except possibly $g_1 = id$ or $g_{t+1} = id$. Also let

$$s' = s \cup \{\langle n, f(n) \rangle\}, \text{ and}$$

$$w_0(s')(l_n) = (l_n), \text{ and}$$

$$w_0(s)(l_n) \text{ is undefined.}$$

Consider the point where $\langle n, f(n) \rangle$ is first used. So we have that

$$w_0 = a x^e b, \text{ and}$$

$$b(s)(l_n) \in \{n, f(n)\},$$

where $a, b \in W_G \cup G$.

If $b$ involves $x$, then either

$$n \in rang(b(s)), \text{ or}$$

$$f(n) \in rang(b(s)), \text{ i.e.,} n \in f^{-1}(rang(b(s))),$$

and so there are only finitely many possibilities for $n$.

Thus without loss of generality, we may assume that $b = g_{t+1}$ in the following.

**Case 1.** Suppose that $n_t > 0$. First suppose that $n_t > 1$. Then we have

$$f(g_{t+1}(l_n)) = f(n).$$

Hence if $f(n) \notin dom(s) \cup \{n\}$, then the computation will stop. Since $dom(s)$ is finite and $f$ is a cofinitary permutation, there are only finitely many possibilities for $n$.

Now assume that $n_t = 1$. To make the computation continue, it has to be that

$$g_t(f(n)) \in dom(s) \cup rang(s) \cup \{n, f(n)\}.$$

Since $g_t$ is a cofinitary permutation and $| f \cap g_t | < \omega$, there are only finitely many $n$'s such that

$$g_t(f(n)) = f(n), or$$

$$g_t(f(n)) = n.$$

Hence there are only finitely many $n$'s to make the computation continue.

So without loss of generality, we may consider that

$$w_0 = g_1 x g_2.$$

Assume that there are infinitely many $l_n$ such that

$$g_1 s' g_2(l_n) = l_n,$$

i.e., $g_1\langle n, f(n)\rangle g_2(l_n) = l_n$. This implies that $g_1 f g_2$ has infinitely many fixed points. Hence

$$| f \cap g_1^{-1} g_2^{-1} | < \omega.$$

This is a contradiction.

**Case 2.** Suppose then $n_t < 0$.

First suppose that $n_t < -1$. We know that if

$$n \notin rang(s) \cup \{f(n)\},$$

the computation stops. There are only finitely many choice for $n$.

So, without loss of generality, we may suppose that $n_t = -1$. Then only

$$g_t(n) \in dom(s) \cup rang(s) \cup \{n, f(n)\}$$

can make the computation continue. Since there are only finitely many $n \in \mathbb{N}$ such that

$$g_t(n) = n, \text{ or}$$

$$g_t(n) = f(n),$$

there are only finitely many possibilities for $n$.

Now we consider the last case

$$w_0 = g_1 x^{-1} g_2.$$

If there are infinitely many $l_n$ such that

$$w_0(l_n) = g_1\langle f(n), n\rangle g_2(l_n) = l_n,$$

then $g_1 f^{-1} g_2$ has infinitely many fixed points. Hence

$$| \{n \in \mathbb{N} \mid f^{-1}(n) = g_1^{-1} g_2^{-2}(n)\} | = \omega,$$

i.e., there is an $g = g_2 g_1 \in G$ such that

$$| g \cap f | = \omega.$$

This is a contradition that $G \cup \{f\}$ is an almost disjoint family in $Sym(\mathbb{N})$.

We have proved the lemma.

□

Now using this lemma, we can easily prove the following lemma.

**Lemma 2.4.** *Let* $G \leq Sym(\mathbb{N})$ *be a cofinitary group and let* $f \in Sym(\mathbb{N}) \setminus G$ *be such that* $G \cup \{f\}$ *is an a.d. family. Get* $g_H$ *be a* $\mathbb{G}_G$*-generic permutation. Then*

$$|g_H \cap f| = \aleph_0$$

*Proof.* We shall force with $\mathbb{G}_G$. Consider the following set

$$C'_{G,n} = \{\langle s, F\rangle \in \mathbb{G}_G \mid \exists m \geq n(f(m) = s(m))\}, \text{ for any } n \in \mathbb{N}.$$

Since $F$ is finite, then by the previous Lemma 2.3, we know that there are only finitely many $\langle n, f(n)\rangle$ such that

$$\langle s \cup \{\langle n, f(n)\rangle\}, F\rangle \not\leq \langle s, F\rangle.$$

Therefore, $C'_{G,m}$ is dense. Thus, if $g_H$ is a $\mathbb{G}_G$–generic permutation, then

$$|g_H \cap f| = \aleph_0$$

□

## 3. Adjoining Cofinitary Permutations.

In this section, we shall provide several forcing models. In each of these models, there exists some maximal cofinitary group $G \leq Sym(\mathbb{N})$ such that $G$ is also a m.a.d. family in $Sym(\mathbb{N})$. We shall prove these results by using the p.o.set $\mathbb{G}_G$.

**Lemma 3.1.** **(MA($\kappa$)).** *Let* $G \leq Sym(\mathbb{N})$ *be a cofinitary permutation group, where* $|G| \leq \kappa$ *and* $\aleph_1 \leq \kappa < 2^{\aleph_0}$*. Assume* $f \in Sym(\mathbb{N}) \setminus G$ *be such that* $G \cup \{f\}$ *is an almost disjoint family. Then there exists a* $g \in Sym(\mathbb{N}) \setminus G$ *such that*

1. $\langle G, g\rangle$ *is a cofinitary group, and*
2. $|f \cap g| = \aleph_0$.

*Proof.* Consider the partial ordering $\mathbb{G}_G$. Consider the following dense subsets of $\mathbb{G}_G$.

$$\begin{aligned} C_{G,w(x)} &= \{\langle s, F\rangle \in \mathbb{G}_G \mid w(x) \in F\}, \text{ for any } w(x) \in W_G, \\ D_{G,n} &= \{\langle s, F\rangle \in \mathbb{G}_G \mid n \in rang(s)\}, \text{ for any } n \in \mathbb{N}, \\ E_{G,n} &= \{\langle s, F\rangle \in \mathbb{G}_G \mid n \in dom(s)\}, \text{ for any } n \in \mathbb{N}, \\ C'_{G,n} &= \{\langle s, F\rangle \in \mathbb{G}_G \mid \exists m \geq n(f(m) = s(m))\}, \text{ for any } n \in \mathbb{N}. \end{aligned}$$

**Note.** The proofs of the density of $D_{G,n}$ and $E_{G,n}$ can be found in [Z1, pp43–47].

Now let

$$\begin{aligned} \mathcal{D} = \{C_{G,w(x)} \mid w(x) \in W_G\} &\cup \{D_{G,n} \mid n \in \mathbb{N}\} \\ \cup\{E_{G,n} \mid n \in \mathbb{N}\} &\cup \{C'_{G,n} \mid n \in \mathbb{N}\}. \end{aligned}$$

It is easy to see that $|\mathcal{D}| \leq \kappa$. By MA($\kappa$), there is a filter $G^*$ in $\mathbb{G}_G$ such that $G^* \cap d \neq \emptyset$ for any $d \in \mathcal{D}$. Then by the density of $D_{G,n}$ and $E_{G,n}$, $n \in \mathbb{N}$, we know that

$$g^* = \cup\{s \mid \exists F \subseteq W_G(\langle s, F\rangle \in G^*)\}$$

is a permutation of $\mathbb{N}$. By the density of $C_{G,w(x)}$ we know that $\langle G, g^*\rangle \leq Sym(\mathbb{N})$ is a cofinitary group (for detailed proof, see e.g. [Z1, Theorem 2.6]). By the density of $C'_{G,n}$ we know that

$$|f \cap g| = \aleph_0.$$

Hence we proved the lemma.

□

**Theorme 3.2.(MA).** *There exists a maximal cofinitary group $G \leq Sym(\mathbb{N})$ such that $G$ is also a maximal almost disjoint family in $Sym(\mathbb{N})$ as well.*

*Proof.* Let $\langle f_\alpha \in Sym(\mathbb{N}) \mid \aleph_0 \leq \alpha < 2^{\aleph_0}\rangle$ be an enumeration of all permutations of $Sym(\mathbb{N})$. We prove the theorem by induction on $\alpha$.

Let $G_n \leq Sym(\mathbb{N})$, $n \in \mathbb{N}$, be any countable cofinitary permutation group. Assume $G_\beta$ has been constructed, where $\aleph_0 \leq \beta < 2^{\aleph_0}$. Let

$$G'_\alpha = \bigcup_{\beta<\alpha} G_\beta.$$

Here $|G'_\alpha| = \kappa < 2^{\aleph_0}$. Consider $\mathbb{G}_{G'_\alpha}$ with $f_\alpha$. We shall construct $g_\alpha$ which satisfies the following conditions.

(1) $g_\alpha \in Sym(\mathbb{N}) \setminus G'_\alpha$ and $\langle G'_\alpha, g_\alpha\rangle$ is cofinitary;
(2) if $|g \cap f_\alpha|$ is finite for any $g \in G'_\alpha$, then $|g_\alpha \cap f_\alpha| = \aleph_0$.

By Lemma 3.1, MA($\kappa$) implies that we can construct such a $g_\alpha$. Let $G_\alpha = \langle G'_\alpha, g_\alpha\rangle$.

Now let $\mathcal{G} = \bigcup_{\alpha<2^{\aleph_0}} G_\alpha$.

By construction, we know that $\mathcal{G} \leq Sym(\mathbb{N})$ is a maximal cofinitary permutation group and $\mathcal{G}$ is also a maximal almost disjoint family in $Sym(\mathbb{N})$.

□

We can also prove that it is consistent with $ZFC$ that there exists a maximal cofinitary group $G \leq Sym(\mathbb{N})$ such that $G$ is a m.a.d. family in $Sym(\mathbb{N})$ and $G$ has cardinality smaller than $2^{\aleph_0}$. We state a well-known lemma about Cohen forcing.

**Lemma 3.3.** *Suppose $I, S \in M$. Let $G$ be $Fn(I,2)$-generic over $M$, and let $X \subseteq S$ with $X \in M[G]$. Then $X \in M[G \cap Fn(I_0,2)]$ for some $I_0 \in M$ and $(|I_0| \leq |S|)^M$.*

*Proof.* See [Kun].

□

The idea of the proof is as follows. Lemma 3.3 implies that it is sufficient to construct in $M$ a maximal cofinitary group $\mathcal{G} \leq Sym(\mathbb{N})$, which is also a maximal almost disjoint family, such that $G$ remains a maximal cofinitary group and a maximal almost disjoint family in any forcing extension $M^{Fn(I_0,2)}$ where $I_0 \in M$ is countable. For any $f \in Sym(\mathbb{N})$ in $M^{Fn(I,2)}$, $f$ is an element of $M^{Fn(I_0,2)}$ for some countable $I_0 \subseteq I$. When $I_0$ is finite, then $M^{Fn(I_0,2)} = M$; and when $|I_0|$ is countable, then $Fn(I_0,2)$ is isomorphic to $Fn(\mathbb{N},2)$ in $M$. Since isomorphic partially ordered sets yields the same generic extension (see e.g. [Kun, p.220]), it is sufficient to construct $\mathcal{G}$ so that whenever $H$ is $Fn(\mathbb{N},2)$-generic over $M$, then there does not exist a permutation $f \in Sym(\mathbb{N}) \cap M[H]$ such that $f$ is a.d. from $g$ for all $g \in \mathcal{G}$.

**Theorem 3.4.** *Let $M \models (ZFC + GCH)$. There is a maximal cofinitary group $\mathcal{G} \leq Sym(\mathbb{N})$ in $M$ of size $\aleph_1$, where $\mathcal{G}$ is also a maximal almost disjoint family of $Sym(\mathbb{N})$ in $M$ such that $\mathcal{G}$ remains maximal with respect to both cofinitary groups and almost disjoint families in $Sym(\mathbb{N})$.*

*Proof.* Since $Fn(\mathbb{N},2)$ has c.c.c. and $M \models GCH$, there are at most $\aleph_0^{\aleph_0} = 2^{\aleph_0} = \aleph_1$ different antichains. Here there are at most $(2^{\aleph_0})^{\aleph_0} = \aleph_1$ nice names for reals. In $M$, we define a maximal cofinitary group $\mathcal{G}$ of size $\aleph_1$ as follows.

Let $\langle p_\alpha, \tau_\alpha\rangle$ for $\aleph_0 \leq \alpha < \omega_1$ enumerate all pairs $\langle p, \tau\rangle$ such that $p \in Fn(\mathbb{N}, 2)$ and $\tau$ is a nice name for a permutation $g_\tau$ of $\mathbb{N}$.

By recursion, we construct an increasing chain of cofinitary groups $G_\alpha$ as follows.

Let $G$ be any countably infinite cofinitary group. We set $G_n = G$ for any $n \in \mathbb{N}$. Assume that $G_\beta \leq Sym(\mathbb{N})$, $\beta < \alpha$, has constructed. Let

$$G'_\alpha = \bigcup_{\beta<\alpha} G_\beta.$$

We choose $f_\alpha$ such that

(1) $\langle G'_\alpha, f_\alpha\rangle \leq Sym(\mathbb{N})$ is a cofinitary group and $f_\alpha \notin G'_\alpha$;
(2) if

$$p_\alpha \Vdash \tau_\alpha \in Sym(\mathbb{N}), \text{ and}$$

$$p_\alpha \Vdash (\tau_\alpha \notin G'_\alpha \text{ and } G'_\alpha \cup \{\tau\} \text{ is almost disjoint family in } Sym(\mathbb{N})),$$

$$\text{then } p_\alpha \Vdash (\tau_\alpha \text{ and } f_\alpha \text{ are not a.d}).$$

To see that $f_\alpha$ maybe so chosen, suppose that the condition of (2) holds.

Consider the p.o.set $\mathbb{G}_{G'_\alpha}$ in $M$ and the dense sets $C_{G'_\alpha, w(x)}, D_{G'_\alpha, n}, E_{G'_\alpha, n}$. And we also consider the set

$$F_{G'_\alpha, n, q} = \{\langle s, F\rangle \in \mathbb{G}_{G', \alpha} \mid \exists x \in \mathbb{N} \exists m \geq n \exists r \geq q$$
$$((s(x) = m) \text{ and } (r \Vdash \tau_\alpha(\dot{x}) = \dot{m}))\}$$

where $q \leq p_\alpha$ and $n \in \mathbb{N}$.

By Lemma 2.3, we can easily prove that $F_{G'_\alpha, n, q}$ is dense in $\mathbb{G}_{G'_\alpha}$.

Now let

$$\mathcal{D} = \{C_{G'_\alpha, w(x)} \mid w(x) \in W_{G'_\alpha}\} \cup \{D_{G'_\alpha, n} \mid n \in \mathbb{N}\} \cup \{E_{G'_\alpha, n} \mid n \in \mathbb{N}\}$$
$$\{F_{G'_\alpha, n, q} \mid n \in \mathbb{N} \text{ and } q \leq p_\alpha\}.$$

Then $|\mathcal{D}| \leq \aleph_0$. By MA($\aleph_0$), there is a filter $H_\alpha \subseteq \mathbb{G}_{G'_\alpha}$ such that $H_\alpha \cap d \neq \emptyset$ for every $d \in \mathcal{D}$. Let

$$f_\alpha = \cup\{s \mid \langle s, F\rangle \in H_\alpha\}.$$

Since $C_{G'_\alpha, w(x)}, D_{G'_\alpha, n}, E_{G'_\alpha, n}$ are all dense, $f_\alpha$ satisfies (1), and since $F_{G'_\alpha, n, q}$ is dense, $f_\alpha$ satisfies (2). Let $G_\alpha = \langle G'_\alpha, f_\alpha\rangle$. Now, let

$$\mathcal{G} = \bigcup_{\alpha<\aleph_1^M} G_\alpha.$$

We claim that $\mathcal{G} \leq Sym(\mathbb{N})$ is a maximal cofinitary group in $M[H]$.

Suppose that $\mathcal{G}$ is not a maximal cofinitary group in $M[H]$. Then there exists a $\langle p_\alpha, \tau_\alpha\rangle$ such that $p_\alpha \in H$ and

$$p_\alpha \Vdash (\tau_\alpha \in Sym(\mathbb{N})), \text{ and}$$

$$p_\alpha \Vdash (\langle \mathcal{G}, \tau_\alpha \in Sym(\mathbb{N}) \text{ is a cofinitary permutation group}).$$

This implies that

$$p_\alpha \Vdash (\mathcal{G} \cup \{\tau_\alpha\} \text{ is an almost disjoint family in } Sym(\mathbb{N})).$$

Thus the condition of (2) holds at $\alpha$, and

$$p_\alpha \Vdash |\tau_\alpha \cap f_\alpha| < \aleph_0.$$

But this contradicts that

$$p_\alpha \Vdash \tau_\alpha \text{ and } f_\alpha \text{ are not a.d..}$$

Thus $\mathcal{G} \leq Sym(\mathbb{N})$ is a maximal cofinitary group in $M[H]$.

Similarly, we can prove that $\mathcal{G}$ is a maximal almost disjoint family in $Sym(\mathbb{N})$.

□

**Theorem 3.5.** *Let* $M \models ZFC + \neg CH$. *Let* $\kappa \in M$ *be a cardinal such that* $\aleph_1 \leq \kappa < 2^{\aleph_0} = \lambda$. *Then, there exists a c.c.c. notion of forcing* $\mathbb{G}$ *such that the following statements holds in* $M^{\mathbb{G}}$.

1. $2^{\aleph_0} = \lambda$.
2. *There exists a maximal cofinitary group* $G \leq Sym(\mathbb{N})$ *of cardinality* $\kappa$ *such that* $G$ *is also a m.a.d. family in* $Sym(\mathbb{N})$ *as well.*

**Note.** The forcing we describe in the following proof of the theorem was introduced in [Z1]. We rewrite it here for the sake of completeness.

*Proof.* By theorem 1.2, there exists a cofinitary group $G \in M$ of cardinality $\kappa$. We define a finite support iterated forcing of length $\omega_1$ as follows.

At the 0th step, we take $G_0 = G$. At step $\alpha$, we assume that we have constructed a cofinitary group $G_\alpha$. At this step, we use the forcing notion $\mathbb{G}_{G_\alpha}$. By Theorem 2.2, we get a new $g_\alpha \in Sym(\omega)$ such that $g_\alpha \notin G_\alpha$ and $G_\alpha * \langle g_\alpha \rangle$ is a cofinitary group $G_{\alpha+1}$. Since $\mathbb{G}_{G_\alpha}$ is a *c.c.c.* forcing, our iterated forcing is *c.c.c.*.

For each $G_\alpha$ there exists a bijection from $W_{G_\alpha}$ onto $\kappa + \alpha$. Then we can take $\mathbb{G}_{G_\alpha}$ to consist of all pairs $\langle s, F \rangle$ where $s$ is a finite 1–1 partial function from $\omega$ to $\omega$, $F$ is a finite subset of $\kappa + \alpha$, and let each $\eta \in F$ stand for the corresponding word. Thus each $\mathbb{G}_{G_\alpha}$ will consist of a set in $M$ (while its partial order is not necessarily in $M$), and the cardinality of $\mathbb{G}_{G_\alpha}$ is therefore $\max(\alpha, \kappa) = \kappa$. Then let $\mathbb{G}$ be the resulting finite support iteration of length $\omega_1$, and let $H \subseteq \mathbb{G}$ be generic over $M$. Since $\mathbb{G}$ is a *c.c.c.* forcing, $\mathbb{G}$ preserves cardinals. Thus $2^\omega$ is the same cardinal in $M[H]$ and $M$.

We claim that $G_{\omega_1}$ is a maximal cofintary permutation group in $Sym(\mathbb{N})$

The proof can be found in [Z1] by using the following lemma, which had been also proved in [Z1].

**Lemma 3.6.** *Let $G$ be a cofinitary permutation group and let $f \in Sym(\mathbb{N}) \setminus G$ be such that $\langle G, f\rangle = G * \langle f\rangle$ is a cofinitary permuation group. Let $g_H$ be a $\mathbb{G}_G$-generic permutation. Then*

$$|f \cap g_H| = \aleph_0.$$

We also claim that $G_{\omega_1}$ is a maximal almost disjoint family in $Sym(\mathbb{N})$ as well.

Assume that $G_{\omega_1}$ is not a maximal almost disjoint family in $Sym(\mathbb{N})$. Let $f \in Sym(\mathbb{N}) \setminus G_{\omega_1}$ in $M[H]$ be such that $G_{\omega_1} \cup \{f\}$ is an almost disjoint family. Let $\dot{f}$ be a nice name of $f$. For each $\langle \dot{n}, \dot{m}\rangle \in \mathbb{N} \times \mathbb{N}$, there exists a maximal antichain $A_{\langle n,m\rangle}$ of $\mathbb{G}$ which decides whether $\dot{f}(n) = m$. Since $\mathbb{G}$ is c.c.c., $A_{\langle n,m\rangle}$ is countable. Let

$$A = \bigcup_{\langle n,m\rangle \in \mathbb{N}\times\mathbb{N}} A_{\langle n,m\rangle}.$$

Then $|A| \leq \aleph_0$. Since $\mathbb{G}$ is an $aleph_1$–length foring with finite support, it is clear that there is a an $\alpha < \omega_1$ such that for any $p \in A$, $supt(p) \subseteq \alpha$. If $H_\alpha$ is the component of $H$ in the iterated forcing up to (but not including) $\alpha$, then we have $f \in M[H_\alpha]$. If $G_\alpha \cup \{f\}$ is an almost disjoint family in $Sym(\mathbb{N})$ and $f \notin G_\alpha$ then Lemma 2.4. implies that

$$|f \cap g_\alpha| = \omega.$$

Hence we completed the proof of the theorem.

□

## 4. Problems.

**Problem 4.1.** *Is it consistent that any maximal cofinitary group is a maximal almost disjoint family in $Sym(\mathbb{N})$?*

**Problem 4.2.** *Let $\mathfrak{a}_\mathfrak{g}$ be the least $\lambda$ such that there exists a maximal cofinitary group $G \leq Sym(\mathbb{N})$ with cardinality $\lambda$. Let $\mathfrak{a}_\mathfrak{p}$ be the least $\lambda$ such that there exists a m.a.d. family $A \subseteq Sym(\mathbb{N})$ with cardinality $\lambda$. Can we prove that it is consistent with ZFC that $\mathfrak{a}_\mathfrak{p} \neq \mathfrak{a}_\mathfrak{g}$?*

**Problem 4.3.** *Let $\mathfrak{a}$ be the least $\lambda$ such that there exists a m.a.d. family $A \subseteq \wp(\mathbb{N})$ with cardinality $\lambda$. Can we prove the consistency of $\mathfrak{a}_\mathfrak{p}$ or $\mathfrak{a}_\mathfrak{g} < \mathfrak{a}$?*

**Problem 4.4.** *Is there any cardinal invariant which is the upper bound of $\mathfrak{a}_\mathfrak{p}$ or $\mathfrak{a}_\mathfrak{g}$?*

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