ADJOINING ALMOST DISJOINT PERMUTATIONS

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Abstract. We show that it is consistent with $ZFC + \neg CH$ that there is a maximal almost disjoint permutation family $A \subseteq Sym(\mathbb{N})$ such that A is a proper subset of an eventually different family $E \subseteq {}^{\mathbb{N}}\mathbb{N}$ and $|A| < |E|$. We also ask several questions in this area.

1. Introduction. We consider two kinds of mathematical structures in this paper: almost disjoint (a.d.) permutation families in $Sym(\mathbb{N})$ and eventually different (e.d.) families of functions in ${}^{\mathbb{N}}\mathbb{N}$.

We say that two permutations $f, g \in Sym(\mathbb{N})$ are a.d. with each other if and only if $|f \cap g| < \omega$, that is, that

$$\{n \in \mathbb{N} \mid f(n) = g(n)\}$$

is finite. And an a.d. permutation family $A \subseteq Sym(\mathbb{N})$ is a subset of $Sym(\mathbb{N})$ such that f, g are a.d. with each other for any $f, g \in A$.

Following E. van Douwen and A. Miller (see, e.g., [vD], [M] and [M1]), we say that two functions $f, g \in {}^{\mathbb{N}}\mathbb{N}$ are e.d. with each other if and only if $|f \cap g| < \omega$. An e.d. family $E \subseteq {}^{\mathbb{N}}\mathbb{N}$ is a subset of ${}^{\mathbb{N}}\mathbb{N}$ such that f, g are e.d. with each other for any $f, g \in E$.

There are several common results about this two families. For example,

(1) Any maximal almost disjoint (m.a.d.) permutation family in $Sym(\mathbb{N})$ has size at least the cardinality of the smallest non-meager set of reals. And same does any maximal eventually different (m.e.d.) families in ${}^{\mathbb{N}}\mathbb{N}$ (see [BSZ] for details).

(2) If we adjoin κ Cohen reals to a ground model of $ZFC + GCH$, then

(2_a) there exists a m.a.d. permutation family $A \subseteq Sym(\mathbb{N})$ such that $|A| = \aleph_1$ (see e.g. [Z3]); and

(2_b) there exists a m.e.d. family in $E \subseteq {}^{\mathbb{N}}\mathbb{N}$ such that $|E| = \aleph_1$ (see, e.g., [Z1], [Z3]).

... etc..

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Note. For more aspects about these two families, the reader can consult [BSZ], [Z] and [Z1]. This paper itself is a continuation of them.

In this paper, we are more interested in finding the difference between these two structures. Of course, it is easily seen that any m.a.d. permutation family $A \subseteq Sym(\mathbb{N})$ is a proper subset of a m.e.d. family in ${}^{\mathbb{N}}\mathbb{N}$. Here we are interested to know the answer of the following question:

Question 1.1. *Is it consistent that there exists a m.a.d. permutation family $A \subseteq Sym(\mathbb{N})$ such that A is a proper subset of a m.e.d. family $E \subseteq {}^{\mathbb{N}}\mathbb{N}$ and $|A| < |E|$?*

We shall give an "yes" answer to this qustion in the second section. In the third section, we shall ask several open questions in this area. The set–theoretical notation that we use in this paper will follow [Jech] or [Kun]. Thus if $\mathbb{P}$ is a notion of forcing and $p, q \in \mathbb{P}$, then $q \leq p$ means that q is a strengthening of p. M always denotes a countable transitive model of ZFC.

2. Adjoining almost disjoint permutations. In this section, we construct a forcing model which can positively answer the Question 1.1. The forcing p. o. set which we will use to construct our model is given in Definition 2.5. We first prove several technical results (Lemma 2.1, and Corollary 2.3) which will be used later on.

Lemma. 2.1.(MA(κ)). *Let $E \subseteq {}^{\mathbb{N}}\mathbb{N}$ with $|E| \leq \kappa < 2^{\aleph_0}$ such that, for any $f \in E$, either $f \in Sym(\mathbb{N})$, or f satisfies the following condition: for any fixed $n \in \mathbb{N}$,*

$$U_n^f = \{2k \in Ev \mid n = f(2k)\}$$

is infinite, where Ev denotes the set of even numbers in $\mathbb{N}$. Then there exists a $g \in {}^{\mathbb{N}}\mathbb{N} \setminus E$ such that

(1) *$g \cap f$ is finite for any $f \in E$, and*
(2) *$U_n^g = \{2k \in Ev \mid g(2k) = n\}$ is infinite for any $n \in \mathbb{N}$.*

Proof. For any $E \subseteq {}^{\mathbb{N}}\mathbb{N}$, let $\mathbb{E}_E$ be the partial order which consists of all conditions of $\langle s, F\rangle$ such that

(a) s is a finite partial function from $\mathbb{N}$ to $\mathbb{N}$, and
(b) F is a finite subset of E.

We define $\langle s_2, F_2\rangle \leq \langle s_1, F_1\rangle$ iff

$$(s_2 \subseteq s_1) \text{ and } (F_2 \subseteq F_1) \text{ and } \forall f \in F_2 (f \cap s_1 \subseteq s_2).$$

Note. The p. o. set $\mathbb{E}_E$ was first introduced by A. Miller in [M]. And, in [Z] and [Z1], several aspects of this p. o. set had been studied. For example, it is easily seen that $\mathbb{E}_E$ fulfils the c.c.c.. Moreover, it can be easily proved that the following sets are dense in $\mathbb{E}_E$,

$$A_n = \{\langle s, F\rangle \in \mathbb{E}_E \mid n \in dom(s) \text{ and } n \in rang(s)\}, \text{ for any } n \in \mathbb{N},$$

$$B_f = \{\langle s, F\rangle \in \mathbb{E}_E \mid f \in F\}, \text{ for any } f \in E.$$

Thus $\mathbb{E}_E$ adjoins a function

$$g = \bigcup\{s \mid \langle s, F\rangle \in G\}, \text{ where } G \text{ is a filter in } \mathbb{E}_E,$$

such that g is e.d. from any $f \in E$.

Now, let $n \in \mathbb{N}$ be any fixed natural number, set

$$C^n_m = \{\langle s, F\rangle \in \mathbb{E}_E \mid \exists 2k \geq m(s(2k) = n)\}, \text{ for any } m \in \mathbb{N}.$$

By the assumption of E, we know that C^n_m is dense in $\mathbb{E}_E$ for any $m, n \in \mathbb{N}$. So we know that the $\mathbb{E}_E$–generic function g satisfies (2) in the lemma, i.e.,

$$U^g_n = \{2k \in Ev \mid g(2k) = n\}$$

is infinite for any $n \in \mathbb{N}$.

Thus let

$$\mathcal{D} = \{A_n \mid n \in \mathbb{N}\} \cup \{B_f \mid f \in E\} \cup \{C^n_m \mid m, n \in \mathbb{N}\}.$$

Since $|\mathcal{D}| \leq \kappa$, by MA($\kappa$), there is a generic filter G^* in $\mathbb{E}_E$ such that

$$G^* \cap d \neq \emptyset, \text{ for any } d \in \mathcal{D},$$

therefore we know that

$$g^* = \bigcup\{s \mid \langle s, F\rangle \in G^*\}$$

satisfies (1) and (2). We hence proved the lemma.

□

The following result was proved by P. Neumann (see e.g., [C, proposition 10.4] for details).

Theorem 2.2.. *There exists an a.d. permutation group* $G \leq Sym(\mathbb{N})$ *such that* $|G| = 2^{\aleph_0}$.

Let $\aleph_1 \leq \kappa < 2^{\aleph_0}$. By P. Neumann's Theorem, we know that there exists an almost disjoint group $G \leq Sym(\mathbb{N})$ with the cardinality κ. It is easily seen that G is also an a.d. family in $Sym(\mathbb{N})$. Thus by repeatedly applying MA, we can prove the following statement.

Corollary 2.3.(MA + ¬CH). *Assume that* $\aleph_1 \leq \kappa < 2^{\aleph_0}$. *Then there exist an a.d. family* $A \subseteq Sym(\mathbb{N})$ *and an e.d. family* $E \subseteq {}^{\mathbb{N}}\mathbb{N}$ *such that*

(1) $A \subseteq E$ *with* $\kappa = |A| < |E| \leq 2^{\aleph_0}$,
(2) *for any* $f \in E \setminus A$ *and for any* $n \in \mathbb{N}$,

$$U^f_n = \{2k \in Ev \mid f(2k) = n\}$$

is infinite.

Definition 2.4. Let $A \subseteq Sym(\mathbb{N})$ be an a.d. family and let $E \subseteq {}^{\mathbb{N}}\mathbb{N}$ be an e.d. family such that $A \subseteq E$ and $A \neq E$. Then the partially ordered set $\mathbb{P}_{A,E}$ consists of all conditions of the form $\langle s, F_A, F_E \rangle$ such that

(1) s is a 1–1 partial function from $\mathbb{N}$ to $\mathbb{N}$;
(2) F_A is a finite subset of A;
(3) F_E is a finite subset of $E \setminus A$.

We define $\langle s_2, F_{A,2}, F_{E,2} \rangle \leq \langle s_1, F_{A,1}, F_{E,1} \rangle$ iff

(a) $s_1 \subseteq s_2$ and $F_{A,1} \subseteq F_{A,2}$ and $F_{E,1} \subseteq F_{E,2}$;
(b) $s_2 \cap f \subseteq s_1$, for any $f \in F_{A,1}$;
(c) $s_2 \cap f \subseteq s_1$, for any $f \in F_{E,1}$.

Notice that since $\langle s_1, F_{A,1}, F_{E,1} \rangle$ and $\langle s_2, F_{A,2}, F_{E,2} \rangle$ are compatible whenever $s_1 = s_2$, $\mathbb{P}_{A,E}$ fulfils the c.c.c.

Lemma 2.5. *Let G be $\mathbb{P}_{A,E}$–generic over M. Then $M[G]$ contains a permutation $g \in Sym(\mathbb{N})$ such that*

(1) *$A \cup \{g\}$ is an a.d. family in $Sym(\mathbb{N})$;*
(2) *$E \cup \{g\}$ is an e.d. family in ${}^{\mathbb{N}}\mathbb{N}$.*

Proof. It is easily seen that the following subsets of $\mathbb{P}_{A,E}$ are dense:

$$B_n = \{\langle s, F_A, F_E \rangle \in \mathbb{P}_{A,E} \mid n \in dom(s)\}, \text{ for any } n \in \mathbb{N};$$
$$C_n = \{\langle s, F_A, F_E \rangle \in \mathbb{P}_{A,E} \mid n \in rang(s)\}, \text{ for any } n \in \mathbb{N};$$
$$D_g = \{\langle s, F_A, F_E \rangle \in \mathbb{P}_{A,E} \mid g \in F_A\}, \text{ for any } g \in A;$$
$$E_f = \{\langle s, F_A, F_E \rangle \in \mathbb{P}_{A,E} \mid f \in F_E\}, \text{ for any } f \in E.$$

If G is $\mathbb{P}_{A,E}$–generic over M, then let

$$g = \bigcup \{s \mid \langle s, F_A, F_E \rangle \in G\}.$$

Hence g satisfies (1) and (2) by construction.

□

Lemma 2.6. *Let A be an a.d. family in $Sym(\mathbb{N})$, let $E \subseteq {}^{\mathbb{N}}\mathbb{N}$ be an e.d. family such that $A \subseteq E$ and for any $f \in E \setminus A$, for any $n \in \mathbb{N}$,*

$$U_n^f = \{2k \in Ev \mid f(2k) = n\}$$

is infinite. If $f \in Sym(\mathbb{N}) \setminus (A \cup E)$ and $A \cup \{f\}$ is an a.d. family in $Sym(\mathbb{N})$, and g is a $\mathbb{P}_{A,E}$–generic permutation, then

$$|f \cap g| = \omega.$$

Proof. Consider the following subset of $\mathbb{P}_{A,E}$:

$$D'_n = \{\langle s, F_A, F_E \rangle \in \mathbb{P}_{A,E} \mid \exists m \geq n (f(m) = s(m))\},$$

for any $n \in \mathbb{N}$. We can prove that D'_n is a dense subset of $\mathbb{P}_{A,E}$. Thus, if g is a $\mathbb{P}_{A,E}$–generic permutation, then $|f \cap g| = \omega$.

□

Theorem 2.7. *Let $M \models (ZFC + MA + \neg CH)$. Let $\kappa, \lambda \in M$ be cardinals such that $\aleph_1 \leq \kappa < 2^{\aleph_0} = \lambda$. Then there exists a c.c.c notion of forcing $\mathbb{P}$ such that the following statement hold in $M^{\mathbb{P}}$.*

(1) $2^{\aleph_0} = \lambda$;
(2) *there exists a mad family $A \subseteq Sym(\mathbb{N})$ of cardinality κ;*
(3) *there exists an e.d. family $E \subseteq {}^{\mathbb{N}}\mathbb{N}$ such that $A \subseteq E$ and $|A| < |E|$.*

Proof. Since $M \models (ZFC + MA + \neg CH)$, then there exist an a.d. family $A \subseteq Sym(\mathbb{N})$ and an e.d. family $E \subseteq {}^{\mathbb{N}}\mathbb{N}$ such that

(a) $A \subseteq E$ with $\kappa = |A| < |E| \leq 2^{\aleph_0}$,
(b) for any $f \in E \setminus A$, for any $n \in \mathbb{N}$,

$$U_n^f = \{2k \in Ev \mid f(2k) = n\}$$

is infinite.

Now we define a finite support iteration forcing $\mathbb{P}$ of length ω_1 as follows.

At the 0th step, we take $G_0 = G$ and $E_0 = E$. At step α, we assume that we have constructed an a.d. permutation family $A_\alpha \subseteq Sym(\mathbb{N})$ and an e.d. family $E_\alpha = A_\alpha \cup E$. At this step, we use the forcing notion $\mathbb{P}_\alpha = \mathbb{P}_{A_\alpha, E_\alpha}$. By Lemma 2.5, we get a new $g_\alpha \in Sym(\mathbb{N})$ such that

(i) $g_\alpha \notin E_\alpha$, and
(ii) $A_{\alpha+1} = A_\alpha \cup \{g_\alpha\}$ is an a.d. family in $Sym(\mathbb{N})$, and
(iii) $E_{\alpha+1} = A_{\alpha+1} \cup E \subseteq {}^{\mathbb{N}}\mathbb{N}$ is an e.d. family.

Since $\mathbb{P}_\alpha$ is a c.c.c. forcing, our iterated forcing is c.c.c.

For each A_α, there exists a bijection from A_α onto $\kappa + \alpha$, and for each $E_\alpha \setminus A_\alpha$, there exists a bijection from $E_\alpha \setminus A_\alpha$ to

$$E^* = \{\beta \in \mathbf{Ord} \mid \kappa + \alpha \leq \beta < \lambda\}.$$

We can take $\mathbb{P}_\alpha$ to consists of all triples $\langle s, F_{A_\alpha}, F_{E_\alpha} \rangle$ where s is a finite 1–1 partial function from $\mathbb{N}$ to $\mathbb{N}$, F_{A_α} is a finite subset of $\kappa + \alpha$ and let $\eta \in F_{A_\alpha}$ stands for the corresponding permutation, and F_{E_α} is a finite subset of E^* and let $\eta \in F_{E_\alpha}$ stands for the corresponding function in ${}^{\mathbb{N}}\mathbb{N}$. Thus each $\mathbb{P}_\alpha$ consists of a set in M (while its partial order is not necessarily in M), and the cardinality of $\mathbb{P}_\alpha$ is λ. Hence $|\mathbb{P}| = \lambda$. Since $\mathbb{P}$ is a c.c.c. forcing, $\mathbb{P}$ preserves cardinals. Thus $2^{\aleph_0}$ is the same cardinal in $M[H]$ and M, where H is $\mathbb{P}$–generic over M.

We claim that $A_{\aleph_1}$ is a m.a.d. permutation family in $Sym(\mathbb{N})$.

Assume not. Let $f \in Sym(\mathbb{N}) \setminus A_{\aleph_1}$ in $M[H]$ be such that $A_{\aleph_1} \cup \{f\}$ is an a.d. permutation family. Let $\dot{f}$ be a nice name of f. For each $\langle n, m \rangle \in \mathbb{N} \times \mathbb{N}$, there exists a maximal antichain $A_{\langle n,m \rangle}$ of $\mathbb{P}$ which decides whether $\dot{f}(n) = m$. Since $\mathbb{P}$ is c.c.c., $A_{\langle n,m \rangle}$ is countable. Let

$$A = \bigcup_{\langle n,m \rangle \in f} A_{\langle n,m \rangle}.$$

Then A is countable. Since $\mathbb{P}$ is an ω_1–length forcing with finite support, it is easily seen that there exists an $\alpha < \omega_1$ such that

$$supp(p) \subset \alpha, \text{ for any } p \in A.$$

If H_α is the component of H in the iterated forcing up to (but not including) α, then we have $f \in M[H_\alpha]$. If $A_\alpha \cup \{f\}$ is an a.d. permutation family in $Sym(\mathbb{N})$, then Lemma 2.6 implies that

$$|f \cap g_\alpha| = \aleph_0.$$

We get contradiction. Thus $A_{\aleph_1}$ is a m.a.d. permutation family in $M[H]$.

Hence we complete the proof of Theorem 2.7.

□

We conclude that it is consistent with $ZFC + \neg CH$ that there exists a m.a.d. permutation family $A \subseteq Sym(\mathbb{N})$ which is a proper subset of an a.d. family $E \subseteq {}^{\mathbb{E}}\mathbb{N}$ such that

$$|A| < |E| \leq 2^{\aleph_0}.$$

3. Some Open Problems.

Problem 3.1. *Let $\mathfrak{a}_\mathfrak{e}$ be the least λ such that there exists a m.e.d family $E \subseteq {}^{\mathbb{N}}\mathbb{N}$ with $|E| = \lambda$. Let $\mathfrak{a}_\mathfrak{p}$ be the least λ such that there exists a m.a.d permutation family $A \subseteq Sym(\mathbb{N})$ with $|A| = \lambda$. Can we prove the consistency of $\mathfrak{a}_\mathfrak{e} \neq \mathfrak{a}_\mathfrak{p}$?*

Problem 3.2. *Let $\mathfrak{a}$ be the least λ such that there exists a m.a.d family $F \subseteq \wp(\mathbb{N})$ with $|F| = \lambda$. We can prove the consistency of $\mathfrak{a} < \mathfrak{a}_\mathfrak{e}, \mathfrak{a}_\mathfrak{p}$ (see e.g. [Z1] and [BSZ]). Can we prove the consistency of $\mathfrak{a}_\mathfrak{e}, \mathfrak{a}_\mathfrak{p} < \mathfrak{a}$?*

Problem 3.3. *Is there any cardinal invariants which is the upper bound of $\mathfrak{a}_\mathfrak{e}$ and $\mathfrak{a}_\mathfrak{p}$?*

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