A CATEGORICAL VERSION OF THE BHK-INTERPRETATION

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A categorical version of the BHK-interpretation

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Abstract

In this paper we interpret (fragments of) intuitionistic logic in categories with weak closure properties, such as weak left exact categories and locally cartesian closed categories (LCCC) with sums. We also interpret the full choice scheme in an LCCC. The interpretation can be seen as a categorical form of the usual Brouwer-Heyting-Kolmogorov (BHK) interpretation. The standard interpretation of geometric logic in a pretopos is obtained by applying the image functor to the BHK-interpretation.

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1 Introduction

The interpretation of a many-sorted first-order language L in a category $\mathbb{C}$ involves assignment of categorical entities to sorts (or types), constants, functions and relations. A sort S is naturally interpreted as an object $\mathcal{M}(S)$ in $\mathbb{C}$, a function $f : S_1 \times \cdots \times S_n \longrightarrow T$ is interpreted as a morphism $\mathcal{M}(f) : \mathcal{M}(S_1) \times \cdots \times \mathcal{M}(S_n) \longrightarrow \mathcal{M}(T)$ in $\mathbb{C}$. A constant c of sort T can be considered as 0-ary function, and thus interpreted as a morphism $\mathcal{M}(c) : 1 \longrightarrow \mathcal{M}(T)$. We shall assume that the category $\mathbb{C}$ has finite products. For the interpretation of relations there seems to be at least two natural choices.

In the standard interpretation of topos theory [8], a relation symbol R on $S_1 \times \cdots \times S_n$ is interpreted as a subobject of $\mathcal{M}(S_1) \times \cdots \times \mathcal{M}(S_n)$, represented by a monomorphism $\mathcal{M}(R) \rightarrowtail \mathcal{M}(S_1) \times \cdots \times \mathcal{M}(S_n)$. An interpretation of a formula φ with free variables of sorts $T_1, \ldots, T_m$ will be a subobject of $\mathcal{M}(T_1) \times \cdots \times \mathcal{M}(T_m)$. We may call this the *propositions-as-subobjects* interpretation.

Another possibility is to interpret the relation as a morphism $f : \mathcal{M}(R) \longrightarrow \mathcal{M}(S_1) \times \cdots \times \mathcal{M}(S_n)$. In this case there may, loosely speaking, be several "reasons" or "proofs" for a tuple to belong to the relation. The interpretation is then extended so that each formula

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φ with free variables of sorts $T_1, \ldots, T_m$ will be a morphism into $\mathcal{M}(T_1) \times \cdots \times \mathcal{M}(T_m)$. This is similar to the propositions-as-types interpretation by Curry and Howard (cf. [9]), where a proposition is the type of its proofs, only that we replace types by categorical objects. This interpretation may be named the *propositions-as-objects* interpretation or the *categorical Brouwer-Heyting-Kolmogorov (BHK) interpretation.*

One important rationale for using internal logic in a category is to facilitate reasoning about the category. The categorical BHK-interpretation is indeed applicable to a wider class of categories than standard (pre)topos-interpretations. For instance to BHK-interpret the $\exists, \wedge$-fragment of intuitionistic logic, i.e. regular logic, it is enough to use a category with finite limits, or even a weak left exact category (see Section 2 and 3). On the other hand, this internal logic is naturally different so that e.g. internal surjections correspond to external split epimorphisms. Nevertheless, it can be shown that for pretoposes the standard interpretation is obtained by applying the image functor to the BHK-interpretation (see Section 4). Weak categories may also be completed so that the standard interpretation becomes applicable, see Carboni [2] or Carboni and Rosolini [3]. In a locally cartesian closed category (LCCC) the BHK-interpretation verifies not only first-order logic but also the full choice scheme (see Section 6 and 7). This is of course not unexpected in view of Seely's interpretation [12] of Martin-Löf type theory [9]. (However [12] is slightly flawed by some subtle coherence problems relating to equality of types. These defects were corrected later by Curien [4] by a change of the type theory, though some constructions of type theory have not been dealt with explicitly.)

The verifications involved in the BHK-interpretation are fairly straightforward and follow the pattern of the standard interpretation [8] quite closely. Hence we give few detailed proofs.

2 Weak Left Exact Categories

The role of pullbacks in categorical interpretations of logic is to do substitution of terms into predicates and formulas. Let $\mathbb{C}$ be a category. A commutative square

$$\begin{tikzcd} D \arrow[r, "q"] \arrow[d, "p"'] & B \arrow[d, "g"] \\ A \arrow[r, "f"'] & C \end{tikzcd} \tag{1}$$

is called a *weak pullback* if for all $p' : E \longrightarrow A$ and $q' : E \longrightarrow B$ such that $f \circ p' = g \circ q'$, there exists a map $r : E \longrightarrow D$ such that $p \circ r = p'$ and $q \circ r = q'$. It is an ordinary (strong) pullback square if, and only if, p and q are jointly monic. A *weak equalizer* is defined similarly to an equalizer, but dropping the uniqueness condition of the universal arrow. The following propositions are proved analogously to their "strong" counterparts.

Proposition 2.1 *A category $\mathbb{C}$ with finite products, has all weak pullbacks if, and only if, it has all weak equalizers.* ∎

Proof. We prove the direction $\Rightarrow$. Suppose that $f, g : A \longrightarrow B$ are morphisms in $\mathbb{C}$. Then a weak equalizer of these is obtained by pulling back the diagonal $\langle 1, 1\rangle : B \longrightarrow B \times B$ along $\langle f, g\rangle : A \longrightarrow B \times B$. ∎

Proposition 2.2 *If the left and right square of the commutative diagram below both are weak pullbacks, then the outer rectangle is a weak pullback.*

$$\begin{tikzcd}
\cdot \arrow[r] \arrow[d] & \cdot \arrow[r] \arrow[d] & \cdot \arrow[d] \\
\cdot \arrow[r] & \cdot \arrow[r] & \cdot
\end{tikzcd}$$

∎

A category is called a *weak left exact category (wlex)* if it has all weak pullbacks and all finite products. A motivation for developing semantics from weak pullbacks is that a natural category arising from type theory (see Example 2.4 below) appears to lack strong pullbacks.

Example 2.3 Let **Grpd** be the category having small groupoids as objects, equivalence classes of functors (between groupoids) as morphisms, and where two functors are identified if they are naturally isomorphic. This category has weak pullbacks and set-indexed products, but in general no pullbacks. We refer to [11] for a proof.

Example 2.4 The category of types in Martin-Löf type theory is an example of a weak left exact category. Its objects are types and the morphisms are functions on types with an equality determined by point-wise by the identity type $\mathrm{Id}_A(a, b)$. More precisely $f : A \longrightarrow B$ and $g : A \longrightarrow B$ are equivalent if the following type is inhabited

$$(\Pi x \in A)\mathrm{Id}_B(f(x), g(x)).$$

The binary product is given by the usual type-theoretic product. The weak pullback of $f : A \longrightarrow C$ and $g : B \longrightarrow C$ is then given by the construction

$$(\Sigma x \in A)(\Sigma y \in B)\mathrm{Id}_C(f(x), g(y)).$$

It is not a pullback object unless uniqueness of identity proofs are assumed, i.e. that each identity type $\mathrm{Id}_D(u, v)$ contains at most one element up to Id-equality. However, we do not know whether this category has pullbacks (in some minimal axiomatization of type theory).

Remark 2.5 An equational presentation of weak pullbacks can be given following Burroni (cf. [6]). For any maps $f : A \longrightarrow C$ and $g : B \longrightarrow C$ there are maps $p_1(f, g) : P(f, g) \longrightarrow A$

and $p_2(f,g) : P(f,g) \longrightarrow B$ such that

$$\begin{tikzcd}
P(f,g) \arrow[r, "p_2(f,g)"] \arrow[d, "p_1(f,g)"'] & B \arrow[d, "g"] \\
A \arrow[r, "f"'] & C
\end{tikzcd} \tag{2}$$

commutes. Moreover, for any $f : A \longrightarrow C$ there is a map $\delta(f) : A \longrightarrow P(f,f)$ such that

$$p_1(f,f)\delta(f) = p_2(f,f)\delta(f) = 1_A. \tag{3}$$

Finally, for any $f : A \longrightarrow C$, $g : B \longrightarrow C$, $h : D \longrightarrow A$ and $k : E \longrightarrow B$ there is a map $\gamma(f,g,h,k) : B(fh,gk) \longrightarrow B(f,g)$ with

$$p_1(f,g)\gamma(f,g,h,k) \;=\; hp_1(fh,gk), \tag{4}$$
$$p_2(f,g)\gamma(f,g,h,k) \;=\; kp_2(fh,gk). \tag{5}$$

3 The Logic of Weak Left Exact Categories

A map $r : R \longrightarrow X$ in a category $\mathbb{C}$ may be regarded as a subobject of X. The intuitive interpretation is that x is in the subobject iff $r^{-1}(x)$ is nonempty. If $s : S \longrightarrow X$ is another map, we say that it is *included in* (R,r) if there is a map $f : S \longrightarrow R$ with $r \circ f = s$. This inclusion relation, denoted $\lesssim$, defines a preorder on the slice category $\mathbb{C}/X$, where $1_X : X \longrightarrow X$ is the terminal object. These maps are equivalent, $(R,r) \approx (S,s)$, if $(R,r) \lesssim (S,s)$ and $(S,s) \lesssim (R,r)$. The corresponding equivalence classes are called *pre-subobjects.* The induced partial order is denoted $(\mathrm{Psub}_{\mathbb{C}}(X), \lesssim)$. Note that weak pullbacks are unique up to pre-subobject equivalence, so any map $f : A \longrightarrow C$ in a wlex category $\mathbb{C}$, defines an order preserving function

$$f^{-1} : \mathrm{Psub}_{\mathbb{C}}(C) \longrightarrow \mathrm{Psub}_{\mathbb{C}}(A)$$

by taking weak pullbacks along f. By Proposition 2.2, it follows that $g^{-1}f^{-1} = (fg)^{-1}$ for $g : X \longrightarrow A$. Consider the weak pullback diagram

$$\begin{tikzcd}
D \arrow[r, "q"] \arrow[d, "p"'] & B \arrow[d, "g"] \\
A \arrow[r, "f"'] & C
\end{tikzcd}$$

Let $r = fp = gq$. For $h : E \longrightarrow C$ we have

$$(E,h) \lesssim (A,f) \text{ and } (E,h) \lesssim (B,g) \text{ if, and only if, } (E,h) \lesssim (D,r).$$

When the weak pullback (D,r) of (A,f) and (B,g) is regarded as a pre-subobject of C, we denote it by $(A,f) \wedge (B,g)$. It follows, similarly as for ordinary subobjects, that inverse functions preserve infima, i.e. for any $k : X \longrightarrow C$,

$$k^{-1}(A \wedge B) \approx k^{-1}(A) \wedge k^{-1}(B).$$

The interpretation $\mathcal{M}$ of the symbols of a many-sorted first-order language is the same as in a left exact category [8] except that a predicate symbol is interpreted as a pre-subobject rather than a subobject. In general, a formula φ with free variables (FV) among $\overline{x} = x_1, \ldots, x_n$ of sorts $S_1, \ldots, S_n$, respectively, will be interpreted as a pre-subobject $(\mathcal{M}_{\overline{x}}(\varphi), m)$ of the product $\mathcal{M}(S_1) \times \cdots \times \mathcal{M}(S_n)$. When the sorts can be inferred from the variables, we also denote the latter product by $\mathcal{M}(x_1, \ldots, x_n)$. A sequent $\varphi_1, \ldots, \varphi_n \overset{\overline{x}}{\Longrightarrow} \psi$ is *(BHK-)valid* in $\mathcal{M}$ iff

$$\mathcal{M}_{\overline{x}}(\varphi_1) \wedge \cdots \wedge \mathcal{M}_{\overline{x}}(\varphi_n) \lesssim \mathcal{M}_{\overline{x}}(\psi).$$

In particular, a sequent $\overset{\overline{x}}{\Longrightarrow} \psi$ is valid in $\mathcal{M}$ iff $1_{\mathcal{M}(\overline{x})} \lesssim \mathcal{M}_{\overline{x}}(\psi)$, i.e. the interpretation map $m : \mathcal{M}_{\overline{x}}(\psi) \longrightarrow \mathcal{M}(\overline{x})$ has a section, i.e. there is some g such that $m \circ g = 1_{\mathcal{M}(\overline{x})}$.

For $FV(\varphi, \psi) \subseteq \overline{x}$, define

$$\mathcal{M}_{\overline{x}}(\varphi \wedge \psi) = \mathcal{M}_{\overline{x}}(\varphi) \wedge \mathcal{M}_{\overline{x}}(\psi).$$

Sequences of formulas are denoted by Γ, Δ etc. The following rules are immediately verified

L1. (Assumption) For $\overline{x} \supseteq FV(\varphi_1, \ldots, \varphi_n)$ and $i = 1, \ldots, n$:

$$\varphi_1, \ldots, \varphi_n \overset{\overline{x}}{\Longrightarrow} \varphi_i$$

L2. (Weakening) For any ψ with $FV(\psi) \subseteq \overline{x}$:

$$\frac{\Gamma \overset{\overline{x}}{\Longrightarrow} \varphi}{\Gamma, \psi \overset{\overline{x}}{\Longrightarrow} \varphi}$$

L3. (Conjunction rules)

$$(\wedge - I)\ \frac{\Gamma \overset{\overline{x}}{\Longrightarrow} \varphi_1 \qquad \Gamma \overset{\overline{x}}{\Longrightarrow} \varphi_2}{\Gamma \overset{\overline{x}}{\Longrightarrow} \varphi_1 \wedge \varphi_2} \qquad\qquad (\wedge - E)\ \frac{\Gamma \overset{\overline{x}}{\Longrightarrow} \varphi_1 \wedge \varphi_2}{\Gamma \overset{\overline{x}}{\Longrightarrow} \varphi_i}\ (i = 1 \text{ or } 2)$$

L4. (Cut rule) For $FV(\Gamma, \Delta, \varphi, \psi) \subseteq \overline{x}$:

$$\frac{\Gamma, \varphi, \Delta \overset{\overline{x}}{\Longrightarrow} \psi \quad \Gamma, \Delta \overset{\overline{x}}{\Longrightarrow} \varphi}{\Gamma, \Delta \overset{\overline{x}}{\Longrightarrow} \psi}$$

Since $\mathbb{C}$ has finite products the interpretation of terms is as usual [8]. If t is a term of sort T, with $FV(t) \subseteq \overline{x} = x_1, \ldots, x_n$, then its interpretation is a morphism $\mathcal{M}_{\overline{x}}(t) : \mathcal{M}(\overline{x}) \longrightarrow \mathcal{M}(T)$, defined by induction on t.

(i) For a variable x_i: $\mathcal{M}_{x_1,\ldots,x_n}(x_i) = \pi_i : \mathcal{M}(x_1, \ldots, x_n) \longrightarrow \mathcal{M}(x_i)$.

(ii) For a constant symbol c of sort T, $\mathcal{M}_{x_1,\ldots,x_n}(c) : \mathcal{M}(x_1, \ldots, x_n) \longrightarrow \mathcal{M}(T)$ is the unique map factoring through $\mathcal{M}(c) : 1 \longrightarrow \mathcal{M}(T)$.

(iii) For a function symbol $f : S_1 \times \cdots \times S_n \longrightarrow T$, and terms $t_1, \ldots, t_n$ of sorts $S_1, \ldots, S_n$, with free variables among $\overline{x}$

$$\mathcal{M}_{\overline{x}}(f(t_1, \ldots, t_n)) = \mathcal{M}(f) \circ \langle \mathcal{M}_{\overline{x}}(t_1), \ldots, \mathcal{M}_{\overline{x}}(t_n) \rangle.$$

Let $m : \mathcal{M}(R) \longrightarrow \mathcal{M}(S_1) \times \cdots \times \mathcal{M}(S_n)$ be an interpretation of the relation symbol R. If $t_1, \ldots, t_n$ are terms of sorts $S_1, \ldots, S_n$, respectively, with free variables among $\overline{x}$, the interpretation of $\mathcal{M}_{\overline{x}}(R(t_1, \ldots, t_n))$ is

$$\langle \mathcal{M}_{\overline{x}}(t_1), \ldots, \mathcal{M}_{\overline{x}}(t_n) \rangle^{-1}(\mathcal{M}(R), m)$$

The equality predicate symbol $=_S$ on a sort S is always interpreted by the diagonal

$$\mathcal{M}(=_S) = \langle 1, 1 \rangle : \mathcal{M}(S) \longrightarrow \mathcal{M}(S) \times \mathcal{M}(S).$$

If the maps $\mathcal{M}_{\overline{x}}(s), \mathcal{M}_{\overline{x}}(t) : \mathcal{M}(S_1) \times \cdots \times \mathcal{M}(S_n) \longrightarrow \mathcal{M}(S)$ are interpretations of terms s and t, then $\mathcal{M}_{\overline{x}}(s = t)$ is the weak equalizer of these maps, according to the proof of Proposition 2.1.

Proposition 3.1 *Let $e : E \longrightarrow X$ be a weak equalizer of $f, g : X \longrightarrow Y$. Then $f = g$ if, and only if, $(X, 1_X) \lesssim (E, e)$.* ∎

Thus the sequent $\overset{\overline{x}}{\Longrightarrow} s = t$ is valid in $\mathcal{M}$ if, and only if, $\mathcal{M}_{\overline{x}}(s) = \mathcal{M}_{\overline{x}}(t)$.

One easily proves that for φ which are formed by conjunctions of atomic formulas:

$$\langle \mathcal{M}_{\overline{u}}(t_1), \ldots, \mathcal{M}_{\overline{u}}(t_n) \rangle^{-1}(\mathcal{M}_{\overline{x}}(\varphi)) \approx \mathcal{M}_{\overline{u}}(\varphi(t_1, \ldots, t_n / x_1, \ldots, x_n)). \qquad (6)$$

From this *substitution equivalence,* the following substitution rule is now easily verified:

L5. (Substitution) For a sequence of terms $\overline{t}$ whose sorts agree with those of the variable sequence $\overline{x}$, item-wise, and for which $FV(\overline{t}) \subseteq \overline{u}$

$$\frac{\Gamma \overset{\overline{x}}{\Longrightarrow} \varphi}{\Gamma(\overline{t}/\overline{x}) \overset{\overline{u}}{\Longrightarrow} \varphi(\overline{t}/\overline{x})}$$

To verify the following equality axioms it is sufficient to use weak equalizers.

L6. (Equality axioms)

$$\xRightarrow{x} x = x, \qquad x = y \xRightarrow{x,y} y = x, \qquad x = y, y = z \xRightarrow{x,y,z} x = z,$$

$$u_1 = v_1, \ldots, u_n = v_n \xRightarrow{\overline{u},\overline{v}} f(u_1, \ldots, u_n) = f(v_1, \ldots, v_n),$$

$$u_1 = v_1, \ldots, u_n = v_n, R(u_1, \ldots, u_n) \xRightarrow{\overline{u},\overline{v}} R(v_1, \ldots, v_n).$$

The existential quantifier is easy to BHK-interpret. For $\alpha : A \longrightarrow X$ and $f : X \longrightarrow Y$ define $\tilde{\exists}_f(A, \alpha) = f \circ \alpha$. Thus we have a well-defined function

$$\tilde{\exists}_f : \mathrm{Psub}(X) \longrightarrow \mathrm{Psub}(Y)$$

which satisfies the following properties. (Note that we do not take the image of $f \circ \alpha$ as in the standard topos-theoretic interpretation.)

Proposition 3.2 *Let $\mathbb{C}$ be a wlex category. For relevant pre-subobjects A and B and maps f, g in $\mathbb{C}$:*

(i) $A \lesssim B \Rightarrow \tilde{\exists}_f(A) \lesssim \tilde{\exists}_f(B)$.

(ii) $\tilde{\exists}_f(A) \lesssim B \Leftrightarrow A \lesssim f^{-1}(B)$.

(iii) $\tilde{\exists}_g \circ \tilde{\exists}_f \approx \tilde{\exists}_{g \circ f}$ *and* $\tilde{\exists}_1 = 1$.

(iv) $\tilde{\exists}_f(f^{-1}(B) \wedge A) \approx B \wedge \tilde{\exists}_f(A)$.

(v) $\tilde{\exists}_{f'}(h^{-1}(A)) \approx g^{-1}(\tilde{\exists}_f(A))$, *if the following is a weak pullback*

$$\begin{tikzcd}
X' \arrow[r, "h"] \arrow[d, "f'"'] & X \arrow[d, "f"] \\
Y' \arrow[r, "g"'] & Y
\end{tikzcd} \quad \blacksquare$$

Proof. (i) and (iii) are immediate by the properties of composition. (ii) is direct by considering the pullback square associated with $f^{-1}(B)$. (iv) and (v) follows from Proposition 2.2 and the fact that weak pullbacks are unique up to $\approx$. ∎

Let φ be a formula with free variables among $\overline{x}, y$, where y has sort S. Then the interpretation of the existential quantifier is given by

$$\mathcal{M}_{\overline{x}}(\exists y \in S.\varphi) =_{\mathrm{def}} \tilde{\exists}_p(\mathcal{M}_{\overline{x},y}(\varphi))$$

where p is the projection $\mathcal{M}(\overline{x}, y) \longrightarrow \mathcal{M}(\overline{x})$. Using Proposition 3.2 it is now straightforward to verify the substitution equivalence (6) for formulas including existential quantifiers as well. The familiar rules for the existential quantifier are straightforward to validate.

L8. (Rules for existential quantification)

$$(\exists I)\ \frac{\Gamma \overset{\overline{x}}{\Longrightarrow} \varphi(b/y)}{\Gamma \overset{\overline{x}}{\Longrightarrow} \exists y \in S.\varphi} \quad (FV(b) \subseteq \overline{x}; b \text{ has sort } S)$$

$$(\exists E)\ \frac{\Gamma \overset{\overline{x}}{\Longrightarrow} \exists y \in S.\varphi \quad \Gamma, \varphi \overset{\overline{x},y}{\Longrightarrow} \psi}{\Gamma \overset{\overline{x}}{\Longrightarrow} \psi} \quad (y \notin \overline{x}, FV(\Gamma, \psi))$$

The rules and axioms of *regular logic* is L1 – L8. We summarize the above claims.

Theorem 3.3 *Regular logic is valid under the BHK-interpretation in a weak left exact category.* ∎

The internal BHK-logic of a wlex category may be used to characterize categorical properties. Most notably, split epimorphisms correspond to surjective functions in this logic.

Proposition 3.4 *Let $\mathbb{C}$ be a wlex category.*

(i) *$f : X \longrightarrow Y$ is a split epi iff this sequent is BHK-valid: $\overset{y}{\Longrightarrow} \exists x \in X.f(x) = y$.*

(ii) *$f : X \longrightarrow Y$ is mono iff this sequent is BHK-valid: $f(x) = f(y) \overset{x,y}{\Longrightarrow} x = y$.*

(iii) *Suppose $e : E \longrightarrow A$ satisfies $f \circ e = g \circ e$, where $f, g : A \longrightarrow B$. Then e is a weak equalizer iff this sequent is BHK-valid:*

$$f(x) = g(x) \overset{x}{\Longrightarrow} \exists z \in E.e(z) = x. \blacksquare$$

4 The Standard Interpretation as an Image of BHK

We show that the standard interpretation of regular logic in a regular category is the image of the BHK-interpretation. In the process we obtain an alternative proof for the basic properties of the standard existential quantifier (cf. Corollary 4.2). Thus one may take BHK as the fundamental interpretation and derive the standard interpretation from it.

Let $\mathbb{C}$ be a category. Denote by $\mathrm{Mon}(\mathbb{C}, X)$ the full subcategory of the slice category $\mathbb{C}/X$ given by the monic arrows into X. There is an inclusion functor

$$U_X : \mathrm{Mon}(\mathbb{C}, X) \longrightarrow \mathbb{C}/X.$$

The category $\mathbb{C}$ is said to have *image factorization* if U_X has a left adjoint Im_X for every object X. It is straightforwardly verified that this condition is equivalent to the following:

for every morphism $\alpha : A \longrightarrow X$ there is a morphism $e_\alpha : A \longrightarrow \mathrm{Im}(\alpha)$ and a monomorphism $m_\alpha : \mathrm{Im}(\alpha) \longrightarrow X$ which factors α, and such for any other factorisation $e : A \longrightarrow C$, $m : C \longrightarrow X$, with m monic, there is a morphism $f : \mathrm{Im}(\alpha) \longrightarrow C$ such that $m \circ f = m_\alpha$.

In a regular category $\mathbb{C}$ image factorization holds and e_α above is a regular epi.

Let $\mathrm{Sub}_{\mathbb{C}}(X)$ denote the subobjects of X in $\mathbb{C}$. The functors U_X and Im_X can be considered as adjoint functors

$$\mathrm{Im}_X : \mathrm{Psub}_{\mathbb{C}}(X) \longrightarrow \mathrm{Sub}_{\mathbb{C}}(X), \qquad \mathrm{U}_X : \mathrm{Sub}_{\mathbb{C}}(X) \longrightarrow \mathrm{Psub}_{\mathbb{C}}(X). \tag{7}$$

Theorem 4.1 *Let $\mathbb{C}$ be a regular category. Then the following holds*

(i) $\mathrm{Im}_X \circ f^{-1} = f^{-1} \circ \mathrm{Im}_Y$ *for* $f : X \longrightarrow Y$*,*

(ii) $\mathrm{Im}_X(\alpha \wedge \beta) = \mathrm{Im}_X(\alpha) \wedge \mathrm{Im}_X(\beta)$*,*

(iii) $\mathrm{Im}_Y \circ \tilde{\exists}_f = \exists_f \circ \mathrm{Im}_X$*, where* $\exists_f = \mathrm{Im}_Y \circ \tilde{\exists}_f$. ■

Proof. In a regular category every epi is a regular epi. Moreover every morphism can be factored as an epi followed by a mono, and this factorisation is unique up to isomorphism. Then (i) follows easily since epi-mono factorisations are preserved by pullbacks. The proof of (ii) uses this fact too, and composition of pullback squares. The proof of (iii) uses simply the uniqueness of epi-mono factorisations. ■

Define $\exists_f = \mathrm{Im}_Y \circ \tilde{\exists}_f$. Now the usual properties of the standard existential quantifier follows from the theorem above, the adjunction (7) and Proposition 3.2.

Corollary 4.2 *For relevant subobjects A and B and maps f, g in a regular category:*

(i) $A \leq B \Rightarrow \exists_f(A) \leq \exists_f(B)$.

(ii) $\exists_f(A) \leq B \Leftrightarrow A \leq f^{-1}(B)$.

(iii) $\exists_g \circ \exists_f = \exists_{g \circ f}$ *and* $\exists_1 = 1$.

(iv) $\exists_f(f^{-1}(B) \wedge A) = B \wedge \exists_f(A)$

(v) $\exists_{f'}(h^{-1}(A)) = g^{-1}(\exists_f(A))$*, if the following is a pullback*

$$\begin{tikzcd}
X' \arrow[r, "h"] \arrow[d, "f'"'] & X \arrow[d, "f"] \\
Y' \arrow[r, "g"'] & Y
\end{tikzcd}$$
■

A *regular formula* is a formula built up from atoms using only $\exists$ and $\wedge$. Theorem 4.1 states that the image operator commutes with substitution, conjunction and existential quantification, so the following result is straightforwardly proven by induction.

Theorem 4.3 *Let $\mathbb{C}$ be a regular category. Suppose that $\mathcal{M}$ is an interpretation where all relation symbols are interpreted as subobjects. Denote by $\widetilde{\mathcal{M}}$ the BHK-interpretation function and let $\overline{\mathcal{M}}$ be the standard interpretation function. Then for regular formulas φ with $FV(\varphi) \subseteq \overline{x}$:*

$$\overline{\mathcal{M}}_{\overline{x}}(\varphi) = \mathrm{Im}(\widetilde{\mathcal{M}}_{\overline{x}}(\varphi)). \blacksquare$$

5 Geometric Logic

We consider the BHK-interpretation of disjunction and extend the results of the previous section.

Recall that a category has finite sums if, and only if, it has binary sums and an initial object 0. Suppose that $\alpha : A \longrightarrow X$ and $\beta : B \longrightarrow X$. Let $[\alpha, \beta] : A + B \longrightarrow X$ be the universal map given by the binary sum. For pre-subobjects $(A, \alpha), (B, \beta), (C, \gamma)$ of X we have

(S1) $(A, \alpha) \lesssim (A + B, [\alpha, \beta])$ and $(B, \beta) \lesssim (A + B, [\alpha, \beta])$,

(S2) If $(A, \alpha) \lesssim (C, \gamma)$ and $(B, \beta) \lesssim (C, \gamma)$, then $(A + B, [\alpha, \beta]) \lesssim (C, \gamma)$.

Sums are *weakly stable* if for any map $f : Y \longrightarrow X$ we have as pre-subobjects of Y:

$$\begin{array}{rcl} f^{-1}(0) & \approx & 0 \\ f^{-1}(A + B) & \approx & f^{-1}(A) + f^{-1}(B). \end{array}$$

The stability condition ensures that substitution works also for formulas involving absurdity and disjunction.

Let $\mathbb{C}$ be a wlex category with finite sums that are weakly stable. Define $\mathcal{M}_{\overline{x}}(\bot)$ to be the unique map $0 \longrightarrow \mathcal{M}(\overline{x})$. Since this is the least element in $\mathrm{Psub}(\mathcal{M}(\overline{x}))$, the absurdity rule is directly verified.

L9. (Absurdity) For any φ with $FV(\varphi) \subseteq \overline{x}$:

$$\frac{\Gamma \overset{\overline{x}}{\Longrightarrow} \bot}{\Gamma \overset{\overline{x}}{\Longrightarrow} \varphi}.$$

Let φ and ψ be formulas with $FV(\varphi, \psi) \subseteq \overline{x}$. Suppose that their interpretations are $\alpha : \mathcal{M}_{\overline{x}}(\varphi) \longrightarrow \mathcal{M}(\overline{x})$ and $\beta : \mathcal{M}_{\overline{x}}(\psi) \longrightarrow \mathcal{M}(\overline{x})$. Then the interpretation of the disjunction is

$$[\alpha, \beta] : \mathcal{M}_{\overline{x}}(\varphi \vee \psi) =_{\mathrm{def}} \mathcal{M}_{\overline{x}}(\varphi) + \mathcal{M}_{\overline{x}}(\psi) \longrightarrow \mathcal{M}(\overline{x}).$$

The following rules are now easily verified using (S1) and (S2).

L10. (Disjunction rules) For $FV(\varphi_1, \varphi_2) \subseteq \overline{x}$:

$$(\vee I)\ \frac{\Gamma \overset{\overline{x}}{\Longrightarrow} \varphi_i}{\Gamma \overset{\overline{x}}{\Longrightarrow} \varphi_1 \vee \varphi_2}\ (i = 1 \text{ or } 2) \qquad (\vee E)\ \frac{\Gamma \overset{\overline{x}}{\Longrightarrow} \varphi_1 \vee \varphi_2 \quad \Gamma, \varphi_1 \overset{\overline{x}}{\Longrightarrow} \psi \quad \Gamma, \varphi_2 \overset{\overline{x}}{\Longrightarrow} \psi}{\Gamma \overset{\overline{x}}{\Longrightarrow} \psi}$$

Geometric logic is defined by the rules L1 – L10. The standard interpretation of this logic in a pretopos is a factor of the abstract BHK-interpretation, since we have the following

Lemma 5.1 *Let $\mathbb{C}$ be a pretopos. Then for pre-subobjects $A \longrightarrow X$ and $B \longrightarrow X$,*

$$\mathrm{Im}(A + B) = \mathrm{Im}(A) \vee \mathrm{Im}(B). \tag{8}$$

Also, $\mathrm{Im}(0 \longrightarrow X) = (0 \longrightarrow X)$.

Proof. Let $\alpha : A \longrightarrow X$ and $\beta : B \longrightarrow X$ be arbitrary morphisms in $\mathbb{C}$. Consider epi-mono factorizations of these maps $e_1 : A \longrightarrow \mathrm{Im}(\alpha)$, $m_1 : \mathrm{Im}(\alpha) \longrightarrow X$ and $e_2 : B \longrightarrow \mathrm{Im}(\beta)$, $m_2 : \mathrm{Im}(\beta) \longrightarrow X$. One easily shows that $e_1 + e_2 : A + B \longrightarrow \mathrm{Im}(\alpha) + \mathrm{Im}(\beta)$ is epi. Then take the epi-mono factorization of $[m_1, m_2]$, which is $e : \mathrm{Im}(\alpha) + \mathrm{Im}(\beta) \longrightarrow \mathrm{Im}([m_1, m_2])$, $m : \mathrm{Im}([m_1, m_2]) \longrightarrow X$. The right hand side of (8) is the subobject m. But $e \circ (e_1 + e_2)$ and m is an epi-mono factorization of $[\alpha, \beta] : A + B \longrightarrow X$. By the uniqueness of such factorizations up isomorphism, the equality (8) follows. ∎

A formula is *geometric* if it is built up from atomic formulas using only $\bot, \wedge, \vee, \exists$. Induction on formulas yields:

Theorem 5.2 *Theorem 4.3 extends to geometric formulas in a pretopos.* ∎

Remark 5.3 Notice the following consequence of this theorem. If a sequent $\varphi \overset{\overline{x}}{\Longrightarrow} \psi$, where φ and ψ are geometric, is BHK-valid in a pretopos, then it is also valid in the standard interpretation. The converse need not hold, in view of Proposition 3.4.(i).

6 First-Order Logic

Now suppose that $\mathbb{C}$ is wlex, has finite sums and that for every $f : X \longrightarrow Y$ there is an order preserving $\forall_f : \mathrm{Psub}(X) \longrightarrow \mathrm{Psub}(Y)$ such that

$$f^{-1}(A) \lesssim B \Leftrightarrow A \lesssim \forall_f(B). \tag{9}$$

Call such a category a *weak hyperdoctrine*. Using the above adjunction (9) one easily proves

Proposition 6.1 *The finite sums of a weak hyperdoctrine are weakly stable.* ∎

The following is a consequence of the adjunction above and the Beck-Chevalley condition for $\tilde{\exists}$.

Proposition 6.2 *In a weak hyperdoctrine,*

$$\forall_{f'}(h^{-1}(A)) \approx g^{-1}(\forall_f(A)),$$

if the following is a weak pullback

$$\begin{tikzcd} X' \arrow[r, "h"] \arrow[d, "f'"'] & X \arrow[d, "f"] \\ Y' \arrow[r, "g"'] & Y \end{tikzcd} \quad \blacksquare$$

For $FV(\varphi) \subseteq \overline{x}, y$ define

$$\mathcal{M}_{\overline{x}}(\forall y \in S.\varphi) =_{\text{def}} \forall_p(\mathcal{M}_{\overline{x},y}(\varphi))$$

where $p : \mathcal{M}(\overline{x}, y) \longrightarrow \mathcal{M}(\overline{x})$ is the obvious projection. For $(B, \beta), (C, \gamma) \in \mathrm{Psub}(X)$ define

$$H((B, \beta), (C, \gamma)) = \forall_\beta((B \wedge C, p)).$$

where p is the projection $B \wedge C \longrightarrow B$. Then it follows that for $(A, \alpha) \in \mathrm{Psub}(X)$,

$$A \wedge B \lesssim C \Longleftrightarrow A \lesssim H(B, C).$$

For $FV(\varphi, \psi) \subseteq \overline{x}$ define

$$\mathcal{M}_{\overline{x}}(\varphi \longrightarrow \psi) = H(\mathcal{M}_{\overline{x}}(\varphi), \mathcal{M}_{\overline{x}}(\psi)).$$

The substitution equalities for $\forall$ and $\longrightarrow$ are consequences of Proposition 6.2. The verification of rules L11 – L22 below follows the usual pattern.

L11. (Rules for universal quantification)

$$(\forall I)\ \frac{\Gamma \overset{\overline{x},y}{\Longrightarrow} \varphi}{\Gamma \overset{\overline{x}}{\Longrightarrow} \forall y \in S.\varphi} \quad (y \notin FV(\Gamma))$$

$$(\forall E)\ \frac{\Gamma \overset{\overline{x}}{\Longrightarrow} \forall y \in S.\varphi}{\Gamma \overset{\overline{x}}{\Longrightarrow} \varphi(b/y)} \quad (FV(b) \subseteq \overline{x}; b \text{ has sort } S)$$

L12. (Implication rules)

$$(\longrightarrow I)\ \frac{\Gamma, \varphi \overset{\overline{x}}{\Longrightarrow} \psi}{\Gamma \overset{\overline{x}}{\Longrightarrow} \varphi \longrightarrow \psi} \qquad (\longrightarrow E)\ \frac{\Gamma \overset{\overline{x}}{\Longrightarrow} \varphi \longrightarrow \psi \quad \Gamma \overset{\overline{x}}{\Longrightarrow} \varphi}{\Gamma \overset{\overline{x}}{\Longrightarrow} \psi}$$

The rules L1 – L12 determine *first-order intuitionistic logic.*

Theorem 6.3 *The rules of first-order intuitionistic logic are BHK-valid in a weak hyperdoctrine.* ∎

Example 6.4 The category of types introduced in Example 2.4 is a weak hyperdoctrine.

7 The Axiom of Choice and LCCC

The logic obtained by a BHK-interpretation into a locally cartesian closed category (LCCC) turns out to be quite powerful. The full choice scheme is valid.

Theorem 7.1 *Any LCCC, with finite sums, is a weak hyperdoctrine.*

Proof. The only verification which is not direct is the existence of the right adjoint $\forall_f$. For this use that Π_f is the right adjoint of f^*. ■

Corollary 7.2 *The rules of first-order intuitionistic logic are BHK-valid in an LCCC with finite sums.* ■

The idea of the following proof is that, under the BHK-interpretation, a proof of a quantifier combination $\forall\exists$ already contains the choice function. See Martin-Löf [9] for a type-theoretic proof.

Theorem 7.3 *In an LCCC with finite sums the following general choice scheme is valid under the BHK-interpretation:*

$$(\forall x \in X)\,(\exists y \in Y)\,\varphi(u,x,y) \overset{u}{\Longrightarrow} (\exists f \in Y^X)\,(\forall x \in X)\,\varphi(u,x,f(x))$$

Proof. Let $r : R \longrightarrow U \times X \times Y$ be the interpretation $\mathcal{M}_{uxy}(\varphi) \longrightarrow U \times X \times Y$. Let $q = \langle \pi_1, \pi_2 \rangle : U \times X \times Y \longrightarrow U \times X$ and $p = \pi_1 : U \times X \longrightarrow U$ be projections. Let $ev : Y^X \times X \longrightarrow Y$ be the evaluation operator. The interpretation $m : \mathcal{M}_{ufx}(\varphi(f(x)/y)) \longrightarrow U \times Y^X \times X$ is the pullback of r along $\varepsilon = \langle \pi_1, \pi_3, ev \circ \langle \pi_2, \pi_3 \rangle \rangle$. Let $\alpha = \langle \pi_1, \pi_2 \rangle : U \times Y^X \times X \longrightarrow U \times Y^X$. Thus $m_1 : \mathcal{M}_u((\forall x \in X)\,(\exists y \in Y)\,\varphi(u,x,y)) \longrightarrow U$ is $\Pi_p \Sigma_q(r) \longrightarrow U$ and $\mathcal{M}_u((\exists f \in Y^X)\,(\forall x \in X)\,\varphi(u,x,f(x))) \longrightarrow U$ is $\Sigma_{p_1}\Pi_\alpha(m) \longrightarrow U$, the composition of $p_2 : \Pi_\alpha(m) \longrightarrow U \times Y^X$ and the first projection $p_1 : U \times Y^X \longrightarrow U$.

We sketch the rest of the proof. Extract the choice function by composing the evaluation $ev' : p^*(\Pi_p \Sigma_q(r)) \longrightarrow \Sigma_q(r)$ with $\pi_3 \circ r : \Sigma_q(r) \longrightarrow Y$. Using that $p^*(\Pi_p \Sigma_q(r))$ is the pullback of m_1 along p one obtains $c : X \times \Pi_p \Sigma_q(r) \longrightarrow Y$ by composing $\pi_3 \circ r \circ ev'$ with the universal map into the pullback. Form the transpose $\hat{c} : \Sigma_q(r) \longrightarrow Y^X$ and then $\langle m_1, \hat{c} \rangle : \Pi_p \Sigma_q(r) \longrightarrow U \times Y^X$. Now it is fairly easy to construct $\psi : X \times \Pi_p(\Sigma_q(r)) \longrightarrow \mathcal{M}_{ufx}(\varphi(f(x)/y)$ such that $m \circ \psi = \langle \langle m_1, \hat{c} \rangle \circ \pi_2, \pi_1 \rangle$. Then since $X \times \Pi_p(\Sigma_q(r))$ is isomorphic to $\alpha^*(\Pi_p(\Sigma_q(r)))$ it follows by the adjunction (α^*, Π_α) that there is a map $\theta : \Pi_p(\Sigma_q(r)) \longrightarrow \Pi_\alpha(m)$ such that $p_2 \circ \theta = \langle m_1, \hat{c} \rangle$. By composition with p_1 we obtain a map $\Pi_p \Sigma_q(r) \longrightarrow \Sigma_{p_1}\Pi_\alpha(m)$ over U as desired. ■

Example 7.4 The category of types with arbitrary equivalence relations [10] in Martin-Löf type theory is an LCCC with finite sums (and a pretopos as well).

Remark 7.5 Notice that the image functor does not commute with implication and universal quantification, since this, together with Theorem 7.3, would imply that AC holds in any topos.

Remark 7.6 Consider the category of sets in ZF set theory. This is a topos and in particular an LCCC. Does this mean that Theorem 7.3 gives a relative consistency proof of choice? Unfortunately not. The principle of excluded middle

$$\overset{\overline{x}}{\Longrightarrow} \varphi \vee \neg\varphi$$

is BHK-valid for all φ if, and only if, ZF proves AC. The direction $\Leftarrow$ is clear. As for $\Rightarrow$, consider a surjective function $f : A \longrightarrow X$. Let R be a unary predicate symbol whose interpretation is this function. Suppose that $\overset{x}{\Longrightarrow} R(x) \vee \neg R(x)$ is BHK-valid. Since $f^{-1}(x)$ is non-empty for each x, it must be the case that all fibers of $\mathcal{M}_x(\neg R(x)) \longrightarrow \mathcal{M}(x)$ are empty. Hence $\overset{x}{\Longrightarrow} R(x)$ is BHK-valid, but this means that f has a section. Since f was arbitrary, AC holds.

Remark 7.7 After a seminar at the Mittag-Leffler institute on March 7, 2001, my attention was drawn to an unpublished paper [1] of Steven Awodey, where he proves a version of Theorem 7.3, relying on Seely [12] for the interpretation of logic. (See also Remark 2.6 in [10].) Moreover, Remark 7.6 is proved more generally for toposes.

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