CONTRACTION-FREE SEQUENT CALCULI FOR GEOMETRIC THEORIES, WITH AN APPLICATION TO BARR'S THEOREIM

S. NEGRI

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INSTITUT MITTAG-LEFFLER
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Contraction-free sequent calculi for geometric theories, with an application to Barr's theorem

Sara Negri

Department of Philosophy
PL 24, Unioninkatu 40 B
00014 University of Helsinki
Finland
negri@helsinki.fi

Abstract

Geometric theories are presented as contraction- and cut-free systems of sequent calculi with mathematical rules following a prescribed rule-scheme that extends the scheme given in Negri and von Plato (1998). As an immediate consequence of cut elimination, it is shown that if a geometric implication is classically derivable from a geometric theory then it is intuitionistically derivable.

Keywords: Cut elimination, geometric theories, Barr's theorem.

Mathematical Subject Classification: 03F05, 18C10, 18B15.

§1. Introduction.

In previous work (Negri and von Plato 1998) it has been shown how to extend sequent calculi with axioms for elementary mathematical theories in such a way that the structural rules (weakening, contraction, and cut) remain admissible for these extensions, by the conversion of axioms to suitably formulated inference rules.

The method of axioms-as-rules permits to convert into cut-free sequent calculi all classical theories axiomatized by (universal closures of) quantifier-free axioms and a wide class of intuitionistic theories with quantifier-free axiomatization, called regular theories.

The method has been applied to predicate logic with equality, theories of apartness and order, affine geometry, and used for proving conservativity, independence, and decidability results (cf. Negri 1999, ch. 6 in Negri and von Plato, Negri, von Plato, and Coquand 2001). Theories such as the theory of ordered fields and real closed fields have also been treated (cf. Negri 2001) by eliminating quantifiers already from their axiomatization through the use of constructions.

In this work we show how to extend the method of axioms-as-rules to geometric theories, i.e., theories axiomatized by (universal closures of) implications between formulas that do not contain $\supset$ or $\forall$.

We introduce the geometric rule-scheme, a rule-scheme that generalizes the regular rule-scheme of Negri and von Plato (1998). Addition of the geometric rule-scheme to the sequent calculi **G3c** and **G3im**, respectively, produces the classical and intuitionistic versions of the given geometric theory.

We show the structural rules admissible for these extensions by a proof that extends the proof in Negri and von Plato (2001) by the use of a suitable substitution lemma in order to handle the restrictions imposed by the variable condition of the geometric rule-scheme.

Examples of geometric theories are given by Robinson arithmetic, the theory of constructive plane affine geometry, the theory of ordered fields and of real closed fields. These examples show the crucial importance of the choice of basic notions for attaining a geometric axiomatization of a theory: For instance a geometric axiomatization of affine geometry requires the notions of apartness, nonincidence, and nonparallelism as primitive in place of their classical counterparts of equality, incidence and parallelism.

In the last section we apply our method to a general result on geometric theories. The result states that if a geometric implication is provable classically in a geometric theory, then it is provable intuitionistically. This result is proved in topos theory by using a completeness theorem for geometric theories in Grothendieck topoi and the construction of a suitable Boolean topos out of a Grothendieck topos.

By our method, the result reduces to a proof-theoretical triviality: A classical proof of a geometric implication in a geometric theory formulated as a sequent system with rules is an intuitionistic proof already. If we add the requirement that the intuitionistic implication does not contain $\perp$ in the antecedent, then the classical proof is indeed a proof in minimal logic.

§2. Preliminaries

We refer to Negri and von Plato (1998,2001) for the necessary background on sequent calculus and its extension with nonlogical (alias mathematical) rules. A brief summary can be found in Section 2 of Negri (2001).

The sequent calculus we shall be using is the contraction- and cut-free sequent calculus **G3**. We list below the rules for its classical version **G3c** and the modifications for obtaining its intuitionistic version **G3im**. The letter "m" stands for multi-succedent.

G3c

Axiom:

$P, \Gamma \Rightarrow \Delta, P$

Logical rules:

$$\frac{A, B, \Gamma \Rightarrow \Delta}{A\&B, \Gamma \Rightarrow \Delta}\ L\& \qquad \frac{\Gamma \Rightarrow \Delta, A \quad \Gamma \Rightarrow \Delta, B}{\Gamma \Rightarrow \Delta, A\&B}\ R\&$$

$$\frac{A, \Gamma \Rightarrow \Delta \quad B, \Gamma \Rightarrow \Delta}{A \vee B, \Gamma \Rightarrow \Delta}\ L\vee \qquad \frac{\Gamma \Rightarrow \Delta, A, B}{\Gamma \Rightarrow \Delta, A \vee B}\ R\vee$$

$$\frac{\Gamma \Rightarrow \Delta, A \quad B, \Gamma \Rightarrow \Delta}{A \supset B, \Gamma \Rightarrow \Delta}\ L\supset \qquad \frac{A, \Gamma \Rightarrow \Delta, B}{\Gamma \Rightarrow \Delta, A \supset B}\ R\supset$$

$$\frac{}{\bot, \Gamma \Rightarrow \Delta}\ L\bot$$

$$\frac{A(t/x), \forall x A, \Gamma \Rightarrow \Delta}{\forall x A, \Gamma \Rightarrow \Delta}\ L\forall \qquad \frac{\Gamma \Rightarrow \Delta, A(y/x)}{\Gamma \Rightarrow \Delta, \forall x A}\ R\forall$$

$$\frac{A(y/x), \Gamma \Rightarrow \Delta}{\exists x A, \Gamma \Rightarrow \Delta}\ L\exists \qquad \frac{\Gamma \Rightarrow \Delta, \exists x A, A(t/x)}{\Gamma \Rightarrow \Delta, \exists x A}\ R\exists$$

G3im

$$\frac{A \supset B, \Gamma \Rightarrow A \quad B, \Gamma \Rightarrow \Delta}{A \supset B, \Gamma \Rightarrow \Delta}\ L\supset \qquad \frac{A, \Gamma \Rightarrow B}{\Gamma \Rightarrow \Delta, A \supset B}\ R\supset$$

$$\frac{\Gamma \Rightarrow A(y/x)}{\Gamma \Rightarrow \Delta, \forall x A}\ R\forall$$

In the axiom, P is an arbitrary atomic formula. Greek upper case Γ, Δ stand for arbitrary multisets of formulas. The restriction in $R\forall$ is that y must not occur free in Γ, (Δ, for **G3c** only), $\forall x A$. The restriction in $L\exists$ is that y must not occur free in $\exists x A, \Gamma, \Delta$. We may summarize these conditions by the requirement that y must not occur free in the conclusion of the two rules.

All the structural rules (weakening, contraction and cut) are admissible in **G3c** and in **G3im**. The calculi are thus complete for classical and intuitionistic first order logic, respectively.

§3. Geometric theories as systems with rules

We recall that a formula in the language of (many-sorted) first-order logic is called *geometric* if it does not contain $\supset$ or $\forall$. A *geometric implication* is a sentence of the form

$$\forall \overline{x}(A \supset B)$$

where A and B are geometric formulas. A *geometric theory* is a theory axiomatized by geometric implications.

As observed in Palmgren (1998), any geometric implication can be reduced to a conjunction of formulas of the form

$$\forall \overline{x}(P_1 \& \dots \& P_m \supset \exists y_1 M_1 \vee \dots \vee \exists y_n M_n) \qquad GA$$

where all the P_i are atomic formulas and all the M_j are conjunctions of atomic formulas and the variables y_j are not free in the P_i. Let M_j be $Q_{j_1} \& \dots \& Q_{j_{k_j}}$ where Q_{j_l} are atomic formulas. We shall call a formula of this form a *geometric axiom*. With a slight abuse of notation, we shall use the vector notation for multisets of formulas (instead of lists) and write $\overline{P}$ for the multiset $P_1, \dots, P_m$ and $\overline{Q}_j$ for $Q_{j_1}, \dots, Q_{j_{k_j}}$. A replacement $\overline{Q}_j(y_j/x_j)$ denotes the replacement in each of the Q_{j_l}, that is, $Q_{j_1}(y_j/x_j), \dots, Q_{j_{k_j}}(y_j/x_j)$.

The rule-scheme corresponding to the geometric axiom GA is

$$\frac{\overline{Q}_1(y_1/x_1), \overline{P}, \Gamma \Rightarrow \Delta \quad \dots \quad \overline{Q}_n(y_n/x_n), \overline{P}, \Gamma \Rightarrow \Delta}{\overline{P}, \Gamma \Rightarrow \Delta} \quad GRS$$

where the variables x_i are called the *replaced variables* of the scheme, and the (lists of) variables y_i the *proper variables* (for better readability we shall leave out the vector notation for lists of variables). The scheme has the condition that the proper variables are not free in $\overline{P}, \Gamma, \Delta$. We shall call a rule-scheme of the above form a *geometric rule-scheme*, GRS for short.

As explained in Negri and von Plato (1998), the *principal formulas* $P_1, \dots, P_m$ of the scheme must be repeated in the antecedent of each premiss for proving the rule of contraction admissible. We also note the following: It can happen that a substitution in the atoms of a rule produces duplications among the formulas P_i. Then in order to ensure that contraction is admissible in the system, we need to add the contracted rule, that is, we must make sure that the following condition is satisfied:

Closure condition: *Given a system with geometric rules, if it has a rule with an instance of form*

$$\frac{\overline{Q}_1(y_1/x_1), P_1, \dots, P_{m-2}, P, P, \Gamma \Rightarrow \Delta \quad \dots \quad \overline{Q}_n(y_n/x_n), P_1, \dots, P_{m-2}, P, P, \Gamma \Rightarrow \Delta}{P_1, \dots, P_{m-2}, P, P, \Gamma \Rightarrow \Delta}$$

then also the rule

$$\frac{\overline{Q}_1(y_1/x_1), P_1, \dots, P_{m-2}, P, \Gamma \Rightarrow \Delta \quad \dots \quad \overline{Q}_n(y_n/x_n), P_1, \dots, P_{m-2}, P, \Gamma \Rightarrow \Delta}{P_1, \dots, P_{m-2}, P, \Gamma \Rightarrow \Delta}$$

has to be included in the system.

The condition is unproblematic, since the number of rules to be added to a given system of nonlogical rules is finite.

Let T be a geometric theory and let **G3cT** (**G3imT**) be the Gentzen system obtained by adding to **G3c**(**G3im**) the geometric rule-schemes corresponding to the geometric axioms of T, together with the rules arising from the closure condition.

A geometric axiom $\Rightarrow A$ (represented as a sequent with empty antecedent) is derivable from the corresponding geometric rule-scheme as follows, where a suffix * denotes repeated steps of a rule, and where the premisses clearly are derivable by $R\&$:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\overline{Q}_1(y_1/x_1),\overline{P}\Rightarrow M_1(y_1/x_1),\dots,M_n(y_n/x_n)}{\overline{Q}_1(y_1/x_1),\overline{P}\Rightarrow \exists x_1M_1,\dots,\exists x_nM_n}{\scriptstyle R\exists^*}\quad\dots\quad\dfrac{\overline{Q}_n(y_n/x_n),\overline{P}\Rightarrow M_1(y_1/x_1),\dots,M_n(y_n/x_n)}{\overline{Q}_n(y_n/x_n),\overline{P}\Rightarrow \exists x_1M_1,\dots,\exists x_nM_n}{\scriptstyle R\exists^*}}{\overline{P}\Rightarrow \exists x_1M_1,\dots,\exists x_nM_n}{\scriptstyle GRS}}{\overline{P}\Rightarrow \exists x_1M_1\vee\dots\vee\exists x_nM_n}{\scriptstyle R\vee^*}}{P_1\&\dots\&P_m\Rightarrow \exists x_1M_1\vee\dots\vee\exists x_nM_n}{\scriptstyle L\&^*}}{\Rightarrow P_1\&\dots\&P_m\supset \exists x_1M_1\vee\dots\vee\exists y_nM_n}{\scriptstyle R\supset}}{\Rightarrow \forall\overline{x}(P_1\&\dots\&P_m\supset \exists x_1M\vee\dots\vee\exists x_nM_n)}{\scriptstyle R\forall}$$

Conversely, a geometric rule is derivable from the corresponding geometric axiom $\Rightarrow A$ in **G3im**+*contr*+*cut* as shown below. In the derivation of the left premiss of cut, *inverses* of rules are used: These are admissible (and height-preserving) steps in **G3im**. Observe that the variable restriction in GRS now comes into use in the steps of $L\exists$:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\Rightarrow\forall\overline{x}(P_1\&\dots\&P_m\supset\exists x_1M_1\vee\dots\vee\exists x_nM_n)}{\Rightarrow P_1\&\dots\&P_m\supset\exists x_1M_1\vee\dots\vee\exists x_nM_n}{\scriptstyle R\forall Inv}}{P_1\&\dots\&P_m\Rightarrow\exists x_1M_1\vee\dots\vee\exists x_nM_n}{\scriptstyle R\supset Inv}}{\overline{P}\Rightarrow\exists x_1M_1\vee\dots\vee\exists x_nM_n}{\scriptstyle L\&Inv}\qquad\dfrac{\dfrac{\dfrac{\overline{Q}_1(y_1/x_1),\overline{P},\Gamma\Rightarrow\Delta}{M_1(y_1/x_1),\overline{P},\Gamma\Rightarrow\Delta}{\scriptstyle L\&^*}}{\exists x_1M_1,\overline{P},\Gamma\Rightarrow\Delta}{\scriptstyle L\exists}\quad\dots\quad\dfrac{\dfrac{\overline{Q}_n(y_n/x_n),\overline{P},\Gamma\Rightarrow\Delta}{M_n(y_n/x_n),\overline{P},\Gamma\Rightarrow\Delta}{\scriptstyle L\&^*}}{\exists x_nM_n,\overline{P},\Gamma\Rightarrow\Delta}{\scriptstyle L\exists}}{\exists x_1M_1\vee\dots\vee\exists x_nM_n,\overline{P},\Gamma\Rightarrow\Delta}{\scriptstyle L\vee^*}}{\overline{P},\overline{P},\Gamma\Rightarrow\Delta}{\scriptstyle cut}}{\overline{P},\Gamma\Rightarrow\Delta}{\scriptstyle contr^*}$$

Remark: It is clear by the above derivation that the geometric rule-scheme hides a cut on the formula $\exists x_1M_1\vee\dots\vee\exists x_nM_n$. The substituted variables x_i are bound variables in the virtual cut formula, so it will be convenient to regard the substituted variables of the scheme as bound variables and to assume that in a derivation the sets of free and bound variables are disjoint.

In order to show that the systems **G3cT** and **G3imT** are complete with respect to the classical and intuitionistic geometric theory T, respectively, we need to extend the results on Negri and von Plato (1998) to systems with geometric rule-scheme.

§4. Admissibility of structural rules

The proofs of (height-preserving) admissibility of structural rules for extensions of sequent calculi with mathematical rules are obtained by induction on the height of derivation. Typically, the rule to be shown admissible is shown to permute up with the rules of the system, until it reaches the axioms which are closed under the rule.

For instance, in order to show that left weakening

$$\frac{\Gamma \Rightarrow \Delta}{A, \Gamma \Rightarrow \Delta}{\scriptstyle LW}$$

is admissible in **G3cT** and in **G3imT**, we consider the last step in the derivation of its premiss, apply inductively weakening to the premiss(es) $\Gamma_i \Rightarrow \Delta_i$ of the last step, and obtain $A, \Gamma_i \Rightarrow \Delta_i$ and then the rule that had been used as last step, thus obtaining $A, \Gamma \Rightarrow \Delta$. However, if the last rule is a geometric rule-scheme and the weakening formula A contains some of its variables, the variable condition is no longer satisfied after weakening with A.

The following lemma takes care of the constraints imposed by the variable condition in the geometric rule scheme in such inductive proofs:

Lemma 1. *Given a derivation of* $\Gamma \Rightarrow \Delta$ *in* **G3cT** (**G3imT**, resp.), *with* x *a free variable in* Γ, Δ *and a term* t *free for* x *in* Γ, Δ *not containing any of the variables of the geometric rules in the derivation, we can find a derivation of* $\Gamma(t/x) \Rightarrow \Delta(t/x)$ *in* **G3cT** (**G3imT**, resp.) *with the same height.*

Proof: By induction on the height of the given derivation. For the logical rules the proof is contained in Lemma 4.1.2 of Negri and von Plato (2001), so we need to consider only the cases arising from the addition of the geometric rule-scheme. Suppose the last rule in the derivation of $\Gamma \Rightarrow \Delta$ is GRS, with premisses

$$\overline{Q}_i(y_i/x_i), \overline{P}, \Gamma' \Rightarrow \Delta$$

for $i = 1, \ldots, n$. Since the $\overline{Q}_i$ are atomic, the term t is free for x in these premisses, thus by induction hypothesis we get derivations of

$$\overline{Q}_i(y_i/x_i)(t/x), \overline{P}(t/x), \Gamma'(t/x) \Rightarrow \Delta(t/x)$$

Since x is a free variable in Γ, Δ, by the remark at the end of the previous section we have $x \neq x_i$, and since the y_i are not free in $\overline{P}, \Gamma, \Delta$ we have $x \neq y_i$. Moreover t does not contain any of the x_i by hypothesis. Therefore the two substitutions in $\overline{Q}_i$ are independent and we have $Q_i(y_i/x_i)(t/x) = Q_i(t/x)(y_i/x_i)$. Since t does not contain any of the y_i, the y_i are not free in $\overline{P}(t/x), \Gamma'(t/x) \Rightarrow \Delta(t/x)$, so we can apply the geometric rule-scheme to the premisses

$$\overline{Q}_i(t/x)(y_i/x_i), \overline{P}(t/x), \Gamma'(t/x) \Rightarrow \Delta(t/x)$$

and get $\overline{P}(t/x), \Gamma'(t/x) \Rightarrow \Delta(t/x)$, i.e., $\Gamma(t/x) \Rightarrow \Delta(t/x)$. □

Theorem 2. *The rules of weakening*

$$\frac{\Gamma \Rightarrow \Delta}{A, \Gamma \Rightarrow \Delta}{\scriptstyle LW} \qquad \frac{\Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, A}{\scriptstyle RW}$$

are admissible and height preserving in **G3cT** *and in* **G3imT**.

Proof: By induction on the height of the derivation of the premiss, as in Negri and von Plato (1998). In case the last step is a geometric rule-scheme and A contains some of its variables, the substitution lemma is applied to the premisses of the geometric rule-scheme in order to have new free variables not clashing with those in A. The conclusion is then obtained by applying the inductive hypothesis and the geometric rule-scheme. □

The proof of admissibility of the contraction rules for **G3cT** and **G3imT** requires the use of inversion lemmas for all those rules that do not copy the principal formula into their premisses. We observe that all the inversion lemmas for the propositional rules that hold for **G3c** and **G3im** hold for their geometric extension as well, since the geometric rule-scheme has only atomic formulas as principal and active formulas. For the inversions of $L\exists$ and $R\forall$ we need to add a condition on the variable in order to avoid clashes with the proper variables of the geometric rules in the derivation.

Let $\vdash_n \ \Gamma \Rightarrow \Delta$ denote a derivation of the sequent $\Gamma \Rightarrow \Delta$ in **G3cT**, with derivation height bounded by n. We have:

Lemma 3.
(i) *If* $\vdash_n \exists x A, \Gamma \Rightarrow \Delta$ *and* y *is not among the variables of the geometric rules in the derivation, then* $\vdash_n A(y/x), \Gamma \Rightarrow \Delta$.
(ii) *If* $\vdash_n \Gamma \Rightarrow \Delta, \forall x A$ *and* y *is not among the variables of the geometric rules in the derivation, then* $\vdash_n \Gamma \Rightarrow \Delta, A(y/x)$.

Proof: (i) By induction on n. If $n = 0$, then $\exists x A, \Gamma \Rightarrow \Delta$ is either a logical axiom, or a conclusion of $L\bot$, or conclusion of a zero-premiss geometrical rule. In each case also $A(y/x), \Gamma \Rightarrow \Delta$ is a logical axiom, or a conclusion of $L\bot$, or conclusion of a zero-premiss geometric rule, thus $\vdash_0 A(y/x), \Gamma \Rightarrow \Delta$.

If $n > 0$ and $\exists x A$ is principal in the last rule, the premiss gives a derivation of $A(z/x), \Gamma \Rightarrow \Delta$, where z is not free in Γ, Δ. By Lemma 1, using the replacement y/z, we obtain a derivation of the same height of $A(y/x), \Gamma \Rightarrow \Delta$. If $\exists x A$ is not principal in the last rule, we argue as in lemma 4.2.3 of Negri and von Plato (2001) in the case the last rule is a logical rule. If the last rule is a geometric rule, with $\Gamma \equiv \overline{P}, \Gamma'$ and premisses

$$\overline{Q}_1(y_1/x_1), \overline{P}, \exists x A, \Gamma' \Rightarrow \Delta, \ldots, \overline{Q}_n(y_n/x_n), \overline{P}, \exists x A, \Gamma' \Rightarrow \Delta$$

by the assumption that free and bound variables are disjoint and by inductive hypothesis we get derivations of the sequents

$$\overline{Q}_1(y_1/x_1), \overline{P}, A(y/x), \Gamma' \Rightarrow \Delta, \ldots, \overline{Q}_n(y_n/x_n), \overline{P}, A(y/x), \Gamma' \Rightarrow \Delta$$

Since y is none of the y_i, we can apply the geometric rule-scheme to these premisses and obtain a derivation of $\overline{P}, A(y/x), \Gamma' \Rightarrow \Delta$.

(ii) Similar to (i). □

We remark that a similar statement holds for **G3imT**, with (ii) modified to an empty context Δ.

By Lemma 1, we are allowed to assume the following:

Disjointness condition. *In a derivation in* **G3cT** (**G3imT**) *the sets of proper variables of the geometric rules are pairwise disjoint.*

Theorem 4. *The rules of contraction*

$$\frac{A, A, \Gamma \Rightarrow \Delta}{A, \Gamma \Rightarrow \Delta}{\scriptstyle LC} \qquad \frac{\Gamma \Rightarrow \Delta, A, A}{\Gamma \Rightarrow \Delta, A}{\scriptstyle RC}$$

are admissible and height-preserving in **G3cT** *and in* **G3imT**.

Proof: For left contraction, the proof is by induction on the height of the derivation of the premiss. If it is an axiom, the conclusion is also an axiom. If the last rule is a propositional rule, then $A, \Gamma \Rightarrow \Delta$ follows as in theorem 3.2 of Negri and von Plato (1998). If it is $L\forall$, we apply the induction hypothesis to the premiss of the rule, and then the rule, and similarly if it is $L\exists$ with A not principal in it. If it is $L\exists$ with $A \equiv \exists x B$ and premiss $B(y/x), \exists x B, \Gamma \Rightarrow \Delta$, by the variable condition on the geometric rule-scheme and the Remark in Section 2, y is not a variable in any geometric rule in the derivation, so we can apply the inversion lemma for $L\exists$ instantiated to y and obtain a derivation of $B(y/x), B(y/x), \Gamma \Rightarrow \Delta$. By the induction hypothesis we get $B(y/x), \Gamma \Rightarrow \Delta$ and by $L\exists$, $\exists x B, \Gamma \Rightarrow \Delta$.

If the last rule is a geometric rule, we distinguish three cases: 1. No occurrence of A is principal in the scheme. 2. One occurrence of A is principal, the other is not. 3. Both occurrences of A are principal.

The first case is handled by a straightforward induction. The second case by the repetition of the principal formulas P_i into the premisses of the geometric-scheme. Finally, the third case is taken care of by the closure condition.

The proof of admissibility of right contraction in **G3cT** and **G3imT** does not present any additional difficulty with respect to the proof of admissibility in **G3c** and **G3im** since in the geometric rule-scheme the succedent in both the premisses and the conclusion is an arbitrary multiset Δ. So in case the last rule in a derivation of $\Gamma \Rightarrow \Delta, A, A$ is a geometric rule, one simply proceeds by applying the inductive hypothesis to the premisses, and then the rule. □

We are now in the position to prove the admissibility of cut for our rule systems for geometric theories. We remark that the proof has the same structure as the proof of admissibility of cut for elementary theories (theorem 6.2.3 in Negri and von Plato 2001), with an additional use of the substitution lemma in order to meet the variable restriction in the geometric rule-scheme.

Theorem 5. *The cut rule*

$$\frac{\Gamma \Rightarrow \Delta, A \quad A, \Gamma' \Rightarrow \Delta'}{\Gamma, \Gamma' \Rightarrow \Delta, \Delta'}\ \mathit{Cut}$$

is admissible in **G3cT** *and in* **G3imT**.

Proof: By induction on the length of A with subinduction on the sum of the heights of the derivations of $\Gamma \Rightarrow \Delta, A$ and $A, \Gamma' \Rightarrow \Delta'$. We need to consider only the cases arising from the addition of the geometric rule-scheme. The other cases are treated in the corresponding proof for **G3c** and **G3im** (theorems 4.2.10 and 5.3.6 in Negri and von Plato 2001).

1. If the left premiss is a zero-premiss geometric rule, then also the conclusion is a zero-premiss geometric rule, since these have an arbitrary context as succedent.

2. If the right premiss is a zero-premiss geometric rule with A not principal in it, the conclusion is a zero-premiss geometric rule for the same reason as in case 1.

3. If the right premiss is a zero-premiss geometric rule with A principal in it, A is atomic and we consider the left premiss. The case that it is a geometric zero-premiss rule is covered by 1. If it is a logical axiom with A not principal, the conclusion is a logical axiom; else Γ contains the atom A and the conclusion follows from the right premiss by weakening. In the remaining cases we consider the last rule in the derivation of $\Gamma \Rightarrow \Delta, A$. Since A is atomic, A is not principal in the rule. Let us consider the case of a geometric rule (the others being dealt with similarly, except $R\supset$ and $R\forall$ which are covered in 4). The derivation ends with

$$\cfrac{\cfrac{\overline{Q}_1(y_1/x_1), \overline{P}, \Gamma'' \Rightarrow \Delta, A \quad \dots \quad \overline{Q}_n(y_n/x_n), \overline{P}, \Gamma'' \Rightarrow \Delta, A}{\overline{P}, \Gamma'' \Rightarrow \Delta, A}\ {\scriptstyle GRS} \qquad A, \Gamma' \Rightarrow \Delta'}{\overline{P}, \Gamma', \Gamma'' \Rightarrow \Delta, \Delta'}\ \mathit{cut}$$

The cut cannot simply be permuted to the premisses of GRS because Γ', Δ' may bring in free variables clashing with the proper variables y_i and thus preventing the application of GRS after the cuts. Instead, we first apply the substitution lemma to the right premiss of cut and replace all the variables y_i (if any) by fresh variables w_i, and denote the substitution by w/y. Observe that by the variable condition in GRS, the substitution does not affect the cut formula A. We obtain the sequent

$$A, \Gamma'(w/y) \Rightarrow \Delta'(w/y)$$

and n cuts with the n premisses of GRS give the n sequents

$$\overline{Q}_i(y_i/x_i), \overline{P}, \Gamma'(w/y), \Gamma'' \Rightarrow \Delta, \Delta'(w/y)$$

for $i = i, \dots n$. By applying GRS to these n premisses we get

$$\overline{P}, \Gamma'(w/y), \Gamma'' \Rightarrow \Delta, \Delta'(w/y)$$

and the derivation is continued as before with the substitution w/y performed globally. Observe that by the disjointness condition the substitution does not affect the active formulas of other geometric rules in the derivation. The cut has thus been replaced by n cuts with left premiss with derivation of lower height and right premiss of same height.

Let us now consider the cases in which neither premiss is an axiom.

4. A is not principal in the left premiss. These are dealt with as above, with cut permuted upwards to the premisses of the last rule used in the derivation of the left premiss (with suitable variable substitution in order to match the variable restrictions in the cases of quantifier rules and geometric rule-scheme), except, in **G3imT**, for $R\supset$ and $R\forall$. By the intuitionistic restriction in this rule, A does not appear in the premiss, and the conclusion is obtained without cut by $R\supset$ ($R\forall$, resp.) and weakening.

5. A is principal in the left premiss only. Then A has to be a compound formula. Therefore, if the last rule of the right premiss is a geometric rule, A cannot be principal in the rule, because only atomic formulas are principal in geometric rules. In this case cut is permuted to the premisses of the right premiss, with appropriate substitution of free variables as in 3, in case the right premiss is a GRS. If the right rule is a logical one with A not principal in it, the usual reductions are applied.

6. A is principal in both premisses. This case can only involve logical rules, and is dealt with as in the usual proof for pure logic. □

Due to the substitutions of free variables, the cut-elimination for derivations of sequents that do not consists only of closed formulas is modulo fresh renaming of free variables.

§5. Examples of geometric theories

(a). A simple example of a geometric theory is *Robinson arithmetic.* The language has one constant 0, the unary successor function s, and two binary functions $+$ and $\cdot$. There is one relation $=$, and atomic formulas are of the form $a = b$, for arbitrary terms a and b. The axioms are the following:

1. $\neg\, s(x) = 0$
2. $s(x) = s(y) \supset x = y$
3. $x = 0 \vee \exists y\, x = s(y)$
4. $x + 0 = x$
5. $x + s(y) = s(x + y)$
6. $x \cdot 0 = 0$

7. $x \cdot s(y) = x \cdot y + x$

The classically equivalent axiomatization with *3.* replaced by

3$'$. $\neg\ x = 0 \supset \exists y\ x = s(y)$

is not geometric because it has an implication $(x = 0 \supset \bot)$ in the antecedent of an implication.

(b). The theory of *nondegenerate ordered fields* has a language with two constants 0 and 1, two functions + and $\cdot$, and one relation $\leqslant$, and the following axioms in which equality is defined by $a = b \equiv a \leqslant b\ \&\ b \leqslant a$:

I. Axioms for nondegenerate linear order

1. $x \leqslant x$
2. $x \leqslant y \vee y \leqslant x$
3. $x \leqslant y\ \&\ y \leqslant z \supset x \leqslant z$
4. $\neg\ 1 \leqslant 0$

II. Axioms for ordered additive group

5. $(x + y) + z = x + (y + z)$
6. $x + y = y + x$
7. $x + 0 = x$
8. $\exists y\ x + y = 0$
9. $x \leqslant y \supset x + z \leqslant y + z$

III. Axioms for multiplication

10. $(x \cdot y) \cdot z = x \cdot (y \cdot z)$
11. $x \cdot y = y \cdot y$
12. $x \cdot 1 = x$
13. $x = 0 \vee \exists y\ x \cdot y = 1$
14. $x \cdot (y + z) = x \cdot y + x \cdot z$
15. $x \leqslant y\ \&\ 0 \leqslant z \supset x \cdot z \leqslant y \cdot z$

The classically equivalent axiomatization with

13$'$. $\neg\ x = 0 \supset \exists y\ x \cdot y = 1$

in place of *13.* is not geometric.

The theory of *real closed fields* is obtained by adding the axioms stating the existence of square roots and zeroes of polynomials of odd degree

16. $0 \leqslant x \supset \exists y\ x = y \cdot y$
17. $a_{2n+1} = 0 \vee \exists x\ a_{2n+1} \cdot x^{2n+1} + a_{2n} \cdot x^{2n} + \ldots a_1 \cdot x + a_0 = 0$

The classically equivalent axiomatization with *17* replaced by

$$17'.\ \neg\, a_{2n+1} = 0 \supset \exists x\ a_{2n+1} \cdot x^{2n+1} + a_{2n} \cdot x^{2n} + \ldots a_1 \cdot x + a_0 = 0$$

is not geometric.

In Negri (2001) the theories of ordered fields and real closed fields, based on a single relation of strict linear order $<$ as primitive, are presented by means of a quantifier-free axiomatization. The $\forall\exists$-axioms expressing the existence of inverses, square roots, and zeroes of polynomials cannot be expressed as geometric axioms by the use of $<$ alone as basic relation. Thus they are replaced by constructions where conditions (like being apart from zero for the inverse) are encoded in the calculus as meta-level rules of well-formedness of terms containing these constructions. The axiomatization thus obtained is geometric and quantifier-free, hence regular.

(c). As last, we give an example of a many-sorted theory, which also shows the crucial importance of the choice of basic notions for having geometric axiomatizations. A quantifier-free axiomatization of *constructive affine geometry*, based on the primitive notions of distinct points, $a \neq b$, distinct lines, $l \neq m$, convergent lines, $l \not\parallel m$, and of a point outside a line, $a \notin l$, is given in 6.6.(e) of Negri and von Plato (2001). The theory has two constructions: of a line $ln(a, b)$ connecting distinct points a and b, and of a point $pt(l, m)$ obtained by the intersection of two nonparallel lines l and m.

One can add to the theory an axiom stating the existence of three non-collinear points,

$$\exists x \exists y \exists z (x \neq y \ \&\ z \notin ln(x, y))$$

and the extension is still a geometric theory. However, if the axiomatization is based instead on the primitive notions of equality between points and between lines, of parallelism of lines, and of incidence of a point with a line, the axiom expressing the existence of three non-collinear points is

$$\exists x \exists y \exists z (\neg\ x = y \ \&\ \neg z \in ln(x, y))$$

and the theory obtained is no longer geometric.

§6. Barr's theorem

As an application of the proof-theoretical method of this paper, we give a proof of a conservativity result for geometric theories.

Barr's theorem is a well known result from topos theory with an important consequence for first order theories (with which it is sometimes identified), stating that if a geometric implication is classically derivable in a geometric theory, then it is intuitionistically derivable. This result is proved in topos theory using a completeness theorem for geometric theories in Grothendieck topoi and a construction–the proper Barr's theorem–of a

Boolean topos out of a Grothendieck topos (cf. Bell 1988, and Johnstone 1977 or Mac Lane and Moerdijk 1992 for a proof in topos theory).

Palmgren (1998) indicates a proof-theoretical proof of Barr's theorem by showing that geometric implications are stable under the Dragalin-Friedman translation. A proof for the special case of the empty geometric theory is suggested in Troelstra and van Dalen, 1988 (exercise 2.6.14) by means of Kripke models.

Using cut-free systems for geometric theories, Barr's theorem reduces to a proof-theoretical triviality:

Theorem 6. *Let T be a geometric theory, and let A be a geometric implication. If* **G3cT** $\vdash\ \Rightarrow A$, *then* **G3imT** $\vdash\ \Rightarrow A$.

Proof: Let A be $\forall\overline{x}(B \supset C)$, and consider a proof of A in **G3cT**. Since B and C do not contain $\supset$ or $\forall$, the derivation of A consists of geometric rules, rules for $\exists$, a step of $R\supset$ and steps of $R\forall$. The geometric rules can occur in any order between the logical rules, however, among the logical rules, $R\supset$ and $R\forall$ come last. The geometric rules have the same succedent in both the premisses and the conclusion, therefore the rules $R\supset$ and $R\forall$ are applied to single-succedent sequents. It follows that the given proof must be a proof in **G3imT** already. □

We recall that *minimal logic* is the logic obtained from intuitionistic logic by removing the rule of *ex falso quodlibet*. In **G3**-calculi, the identity axiom scheme is restricted to atomic formulas, thus in order to obtain a **G3** calculus for minimal logic the rule $\bot E$ is removed, but its instance $\bot Ax$, $\bot, \Gamma \Rightarrow \Delta, \bot$ is added (usually this is just an instance of the unrestricted identity axiom). Let **G3mm** be the calculus thus obtained.

If we assume that $\bot$ does not appear in the antecedent of a geometric implication (though it may appear in the succedent, in the form of an empty disjunction), a (classical) proof of a geometric implication in a geometric theory never uses the rule $\bot E$, neither can it start with logical axioms of the form $\bot, \Gamma \Rightarrow \Delta, \bot$. Under this hypothesis, Theorem 6 can thus be strengthened to

Corollary 7. *Let T be a geometric theory, and let A be a geometric implication not containing $\bot$ in the antecedent. If* **G3cT** $\vdash\ \Rightarrow A$, *then* **G3mmT**$-\bot Ax\ \vdash\ \Rightarrow A$.

§7. Concluding remarks

As explained in Section 6.3 of Negri and von Plato (2001), there are other approaches to extension by axioms, besides systems with mathematical rules.

For quantifier-free regular theories, as those considered in Negri and von Plato (1998), one can add "mathematical basic sequents" of the form

$$P_1, \ldots, P_m \Rightarrow Q_1, \ldots, Q_n$$

with which derivations can start. Cuts in a derivation are then reduced by the generalized Hauptsatz to cuts on atoms.

For geometric theories the axioms cannot be reduced to their atomic components, and the basic sequents to be added have the form

$$P_1, \ldots, P_m \Rightarrow \exists x_1 M_1, \ldots, \exists x_n M_n$$

where M_j are conjunctions of atoms. It can then be shown that cuts can be reduced to cuts on the formulas P_i and $\exists x_j M_j$. As remarked by Thomas Strahm, this partial cut-elimination result would suffice for a proof of Barr's theorem of the kind we have given here.

However, the proof of partial cut-elimination requires permutation of cuts: In a typical critical case, a cut on $\exists x_j M_j$, the premisses of which are a mathematical axiom and a step of $L\exists$, is followed by a cut on an arbitrary formula A. The cut on $\exists x_j M_j$ cannot be further reduced, and the cut on A cannot simply be permuted above it, since it may bring in variables clashing with the free variable in the premiss of $L\exists$. So a substitution lemma of the kind we have employed in our proof comes into play. When all details have been spelled out, the proof of the partial cut-elimination theorem is not simpler than the full cut-elimination.

The theories of real closed fields and formally real fields, together with some other geometric theories, have been treated in the work of Coste, Lombardi, and Roy (2001) by what they call the method of *dynamical proof* : such proofs provide a uniform method to deal with geometric theories, which is related to cut-free sequent calculus. In contrast to sequent calculi, however, dynamical proofs do not presuppose any choice of logic (classical or intuitionistic), since they act upon atomic formulas and $\perp$ only, exactly like proofs of basic sequents in (cut-free) systems with mathematical rules.

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