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A Dichotomy in the Complexity of Propositional Circumscription

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Abstract

The inference problem for propositional circumscription is known to be highly intractable and, in fact, harder than the inference problem for classical propositional logic. More precisely, in its full generality this problem is $\Pi_2^{\rm P}$*-complete, which means that it has the same inherent computational complexity as the satisfiability problem for quantified Boolean formulas with two alternations (universal-existential) of quantifiers. We use Schaefer's framework of generalized satisfiability problems to study the family of all restricted cases of the inference problem for propositional circumscription. Our main result yields a complete classification of the "truly hard" (*$\Pi_2^{\rm P}$*-complete) and the "easier" cases of this problem (reducible to the inference problem for classical propositional logic). Specifically, we establish a dichotomy theorem which asserts that each such restricted case either is* $\Pi_2^{\rm P}$*-complete or is in* coNP*. Moreover, we provide efficiently checkable criteria that tell apart the "truly hard" cases from the "easier" ones.*

1 Introduction and Summary of Results

During the past three decades, researchers in artificial intelligence have investigated in depth various formalisms of nonmonotonic reasoning. Circumscription, introduced by McCarthy [McC80], is perhaps the most well-known and extensively studied such formalism. It enjoys high expressive power and thus is suitable for modeling a wide variety of problems requiring nonmonotonic reasoning. Moreover, propositional circumscription has been shown by Gelfond et al. [GPP89] to coincide with reasoning under the extended closed world assumption (ECWA), which is one of the main formalisms for reasoning with incomplete information.

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A fundamental problem in every logical formalism is *inference,* i.e., the problem of deciding whether, given two formulas φ and ψ, the formula ψ can be inferred from φ in the context of the logical formalism at hand. Intuitively, φ represents a knowledge base, while ψ represents a statement that we are interested in deciding whether it can be inferred from the knowledge base. In the case of classical propositional logic, inference amounts to *tautological implication* $\varphi \models \psi$, i.e., to the problem of deciding whether ψ is satisfied by every truth assignment that satisfies φ. Consequently, inference in classical propositional logic is a coNP-complete problem and thus considered to be intractable. In the case of propositional circumscription, inference turns out to have even higher inherent computational complexity. Indeed, as shown by Eiter and Gottlob [EG93], the inference problem for propositional circumscription is $\Pi_2^{\rm P}$-complete. Recall that the class $\Pi_2^{\rm P}$ constitutes the second level of the polynomial hierarchy PH and thus contains both NP and coNP as subclasses. Moreover, the prototypical $\Pi_2^{\rm P}$-complete problem is $\Pi_2^{\rm P}$-SAT, i.e., the satisfiability problem for quantified Boolean formulas of the form $\forall\bar{x}\exists\bar{y}\theta(\bar{x},\bar{y})$, where $\bar{x}$, $\bar{y}$ are tuples of propositional variables and $\theta(\bar{x},\bar{y})$ is a CNF-formula (see [Pap94]).

Classical propositional logic is concerned with all models of a given formula, i.e., with all truth assignments that satisfy the formula. In contrast, propositional circumscription is concerned with the *minimal* models of a given formula, i.e., with those satisfying truth assignments for which there is no smaller satisfying truth assignment with respect to the coordinate-wise partial order between truth assignments. Consequently, in its full generality, the inference problem for propositional circumscription can be stated as follows: given two CNF-formulas φ and ψ, is ψ true in every minimal model of φ? A moment's reflection reveals that this problem is polynomial-time equivalent to the special case in which ψ is simply a clause (i.e., a disjunction of literals), since ψ can be inferred from φ under propositional circumscription if and only if each clause of ψ can be so inferred. Moreover, Eiter and Gottlob [EG93] established that the inference problem for propositional circumscription remains $\Pi_2^{\rm P}$-complete even when φ is a 3CNF-formula and the clause ψ consists of a single negated variable.

Are there restricted classes of propositional formulas on which the inference problem for propositional circumscription has complexity lower than $\Pi_2^{\rm P}$-complete? To make this question precise, one can consider restrictions on both the formulas representing knowledge bases and the formulas representing statements to be inferred. Since clauses are the syntactically simplest propositional formulas, it is natural to consider restrictions on the formulas representing knowledge bases only. Thus, for every class $\mathcal{F}$ of propositional formulas, we let INF-CIRC($\mathcal{F}$) denote the following decision problem: given a formula $\varphi \in \mathcal{F}$ and a clause ψ, is ψ true on every minimal model of φ? The question then is to analyze the computational complexity of INF-CIRC($\mathcal{F}$) for different classes $\mathcal{F}$ of propositional formulas and identify classes $\mathcal{F}$ for which the complexity of INF-CIRC($\mathcal{F}$) is lower than $\Pi_2^{\rm P}$-complete. Even before the $\Pi_2^{\rm P}$-completeness of the full problem was established, this question was studied by Cadoli and Lenzerini [CL94], where INF-CIRC($\mathcal{F}$) was shown to be in P or to be coNP-complete for several different classes $\mathcal{F}$ of propositional formulas. Specifically, Cadoli and Lenzerini observed that if a class $\mathcal{F}$ of propositional formulas is such that testing satisfying truth assignments for minimality is in polynomial time, then INF-CIRC($\mathcal{F}$) is in coNP. Since minimality testing is in polynomial time for the classes of Horn formulas, dual Horn formulas and 2CNF-formulas, it follows that INF-CIRC($\mathcal{F}$) is in coNP, when $\mathcal{F}$ is one of these three classes. Moreover, if $\mathcal{F}$ is the class of all Horn formulas, then INF-CIRC($\mathcal{F}$) is solvable in polynomial time, since every satisfiable Horn formula has a minimum (unique minimal) model that can be computed in polynomial time. In [CL94], it was also proved that INF-CIRC($\mathcal{F}$) is actually coNP-complete, when $\mathcal{F}$ is the class of all dual Horn formulas or the class of all 2CNF-formulas.

The aforementioned results identify several interesting cases where the complexity of the inference problem in propositional circumscription is lower than $\Pi_2^{\rm P}$-complete. Nonetheless, they do not provide a *complete* classification of the "truly hard" ($\Pi_2^{\rm P}$-complete) and the "easier" cases of this problem. In particular, except for the class of all CNF-formulas and the class of all 3CNF-formulas, no other interesting classes $\mathcal{F}$ of propositional formulas for which INF-CIRC($\mathcal{F}$) is $\Pi_2^{\rm P}$-complete were known prior to the work reported here. This should be contrasted with the state of affairs concerning the complexity of the inference problem for classical propositional logic, where a complete classification can be derived from the pioneering work by Schaefer [Sch78] on the complexity of GENERALIZED SATISFIABILITY problems. In order to describe Schaefer's work and relate it to the inference problem, we need to introduce some terminology and notation.

A *logical relation* (or *generalized connective*) R is a non-empty subset of $\{0,1\}^k$, for some $k \geq 1$. If $S = \{R_1, \ldots, R_m, \ldots\}$ is a set of logical relations, then an $\mathcal{F}(S)$-formula is a conjunction of expressions (called *generalized clauses* or, simply, *clauses*) of the form $\mathbf{R}_i(x_1, \ldots, x_k)$, where each $\mathbf{R}_i$ is a relation symbol representing the logical relation R_i in S and each x_j is a Boolean variable. Furthermore, an $\mathcal{F}_{\rm C}(S)$-formula is a formula obtained from an $\mathcal{F}(S)$-formula by substituting some of the variables by the constant symbols $\mathbf{0}$ and $\mathbf{1}$. Each set S of logical relations gives rise to the following GENERALIZED SATISFIABILITY problem SAT$_{\rm C}(S)$: given an $\mathcal{F}_{\rm C}(S)$-formula φ, is φ satisfiable? In a similar manner, one obtains the family of SAT(S) problems by considering $\mathcal{F}(S)$-formulas, instead of $\mathcal{F}_{\rm C}(S)$-formulas.

In [Sch78], four conditions were isolated and the following remarkable classification theorem for the family of all GENERALIZED SATISFIABILITY problems SAT$_{\rm C}(S)$ was established: if the set S satisfies at least one of these four conditions, then SAT$_{\rm C}(S)$ is solvable in polynomial time; otherwise, SAT$_{\rm C}(S)$ is NP-complete. These four conditions are: (1) every relation in S is the set of models of a Horn formula; (2) every relation in S is the set of models of a dual Horn formula; (3) every relation in S is the set of models of a 2CNF formula; (4) every relation in S is the set of models of an affine formula, i.e., a conjunction of formulas built using the $\oplus$ (exclusive or) connective. It should be noted that each of these conditions turned out to be efficiently checkable. Schaefer also obtained a classification theorem for the family of SAT(S) problems, which involves two additional conditions that trivially give rise to polynomial-time solvable SAT(S) problems. Note that the NP-completeness of POSITIVE 1-IN-3-SAT, NOT-ALL-EQUAL 3-SAT and other well known variants of SAT is an immediate consequence of Schaefer's results. Morover, the above results constitute the first instance of a *dichotomy theorem* for a family of decision problems in NP, i.e., results that concern an infinite family $\mathcal{C}$ of decision problems and assert that certain problems in $\mathcal{C}$ are NP-complete, while on the contrary all other problems in $\mathcal{C}$ are solvable in polynomial time. It should be pointed out that the a priori existence of dichotomy theorems cannot be taken for granted, since Ladner's theorem in [Lad75] asserts that if P $\neq$ NP, then there are problems in NP that are neither NP-complete nor in P.

The inference problem in classical propositional logic is polynomial-time reducible to the satisfiability problem. Using this fact, it is easy to see that Schaefer's dichotomy theorem for satisfiability problems yields a dichotomy theorem for the inference problem in classical propositional logic. Specifically, if S is a set of logical relations that satisfy at least one of the four aforementioned conditions, then the inference problem in classical propositional logic for $\mathcal{F}_{\rm C}(S)$-formulas is solvable in polynomial time; otherwise, it is coNP-complete. In addition, a similar dichotomy theorem can be derived for the inference problem in classical propositional logic for $\mathcal{F}(S)$-formulas.

In this paper, we use Schaefer's framework to investigate the computational complexity of the inference problem in propositional circumscription. Our main result asserts that, for every set S of logical relations, either INF-CIRC$(\mathcal{F}_{\rm C}(S))$ is $\Pi_2^{\rm P}$-complete or INF-CIRC$(\mathcal{F}_{\rm C}(S))$ is in coNP. In other words, our main result tells that each restricted cases of the inference problem for propositional circumscription either is as hard as the general case or is reducible to the inference problem for classical propositional logic. Moreover, it provides efficiently checkable criteria that, given a finite set S of logical relations, distinguish the two possibilities for the complexity of INF-CIRC$(\mathcal{F}_{\rm C}(S))$. This constitutes a dichotomy theorem for the inference problem in propositional circumscription, since results by Ladner [Lad75] imply that if $\Pi_2^{\rm P} \neq$ coNP, then there are decision problems in $\Pi_2^{\rm P}$ that are neither $\Pi_2^{\rm P}$-complete nor in coNP. It should also be pointed out that the boundary in the dichotomy separating $\Pi_2^{\rm P}$-completeness from membership in coNP turns out to be different from the boundary in the dichotomy theorem for the inference problem in classical propositional logic.

Our main result is established in two stages. In the first stage, we prove a dichotomy theorem for the family of INF-CIRC$(\mathcal{F}_{\rm C}(S))$ problems, where S is a set of 1-*valid* logical relations, i.e., each relation in S contains the all-ones tuple $(1,\ldots,1)$. In the second stage, we use this restricted dichotomy theorem as a stepping stone to derive the dichotomy theorem for the full family of INF-CIRC$(\mathcal{F}_{\rm C}(S))$ problems, where S is an arbitrary set of logical relations. To this effect, we apply the restricted dichotomy theorem to the set S^* of all 1-valid logical relations obtained from relations in S by replacing some variables by 0. A two-stage approach was used for the first time in a recent paper [KK01], where a dichotomy theorem for minimal satisfiability problems was established. With some extra work, we can also obtain a dichotomy theorem for the family of all INF-CIRC$(\mathcal{F}(S))$ problems, where S is a set of logical relations. Due to space limitations, this result will be presented in the full version of the present paper.

Since the publication of the original dichotomy theorem by Schaefer [Sch78], researchers have obtained several other dichotomy theorems for certain variants of satisfiability problems (see, for instance, [Cre95, KSW97, CH96, CH97, KS98, RV00, KK01]). The results reported here provide the first dichotomy between $\Pi_2^{\rm P}$-completeness and membership in coNP. At the technical level, the proofs make extensive use of Schaefer's expressibility theorem [Sch78, Theore 3.0], as well as of a definability result by Creignou and Hébrard [CH97] and other special-purpose definability results established here.

Finally, we conjecture that a *trichotomy* theorem holds for the complexity of propositional circumscription. Specifically, we conjecture that, for every set S of logical relations, exactly one of the following three alternatives holds: (1) INF-CIRC$(\mathcal{F}_{\rm C}(S))$ is $\Pi_2^{\rm P}$-complete; (2) INF-CIRC$(\mathcal{F}_{\rm C}(S))$ is coNP-complete; (3) INF-CIRC$(\mathcal{F}_{\rm C}(S))$ is solvable in polynomial time. Note that if this conjecture is confirmed, it will yield the first trichotomy theorem for a family of natural decision problems in a complexity class beyond NP. In view of the dichotomy theorem established here, it remains to establish a dichotomy theorem for those INF-CIRC$(\mathcal{F}_{\rm C}(S))$ problems that are in coNP. Although the results in [CL94] yield parts of this conjectured dichotomy, much more remains to be done in order to complete the picture.

2 Preliminaries and Background

This section contains a minimum amount of the necessary background material on the complexity of GENERALIZED SATISFIABILITY problems from [Sch78].

Let $S = \{R_1,\ldots,R_m,\ldots\}$ be a set of logical relations of various arities. As stated in Section 1, an $\mathcal{F}(S)$-formula is a finite conjunction of clauses built using relations from S and propositional variables, while an $\mathcal{F}_{\rm C}(S)$-formula is a formula built using relations from S, propositional variables, and the constant symbols **0** or **1**. Recall also that SAT(S) is the following decision problem: given an $\mathcal{F}(S)$-formula φ, is it satisfiable? (i.e., is there a truth assignment to the variables of φ that makes every clause of φ true?) The decision problem SAT$_{\rm C}(S)$ is defined in a similar way.

Clearly, for each finite set S of logical relations, both SAT(S) and SAT$_{\rm C}(S)$ are problems in NP. Several well-known NP-complete problems can easily be cast as SAT(S) problems for particular sets S of logical relations. For example, 3-SAT coincides with the problem SAT(S), where $S = \{R_0, R_1, R_2, R_3\}$ and $R_0 = \{0,1\}^3 - \{(0,0,0)\}$ (expressing the clause $(x \vee y \vee z)$), $R_1 = \{0,1\}^3 - \{(1,0,0)\}$ (expressing the clause $(\neg x \vee y \vee z)$), $R_2 = \{0,1\}^3 - \{(1,1,0)\}$ (expressing the clause $(\neg x \vee \neg y \vee z)$), and $R_3 = \{0,1\}^3 - \{(1,1,1)\}$ (expressing the clause $(\neg x \vee \neg y \vee \neg z)$). Similarly, the NP-complete problem POSITIVE-1-IN-3-SAT ([GJ79, LO4, page 259]) is precisely the problem SAT(S), where S is the singleton consisting of the relation $R_{1/3} = \{(1,0,0),(0,1,0),(0,0,1)\}$.

Recall that a *Horn* formula is a conjunction of clauses each of which is a disjunction of literals such that at most one of them is a variable. Similarly, a *dual Horn* formula is a conjunction of clauses each of which is disjunction of literals such that at most one of them is a negated variable. As mentioned in Section 1, an *affine* formula is a conjunction of subformulas each of which is an *exclusive disjunction* $\oplus$ of literals or a negation of an exclusive disjunction of literals.

Definition 2.1: Let R be a logical relation and S a finite set of logical relations.

R is 1-*valid* if it contains the tuple $(1, 1, \ldots, 1)$, whereas R is 0-*valid* if it contains the tuple $(0, 0, \ldots, 0)$. We say that S is 1-*valid* (0-*valid*) if every member of S is 1-valid (0-valid).

R is *2CNF* (*Horn, dual Horn,* or *affine,* respectively) if there is a propositional formula φ which is 2CNF (Horn, dual Horn, or affine, respectively) and such that R coincides with the set of truth assignments satisfying φ.

S is *Schaefer* if at least one of the following four conditions hold: every member of S is 2CNF; every member of S is Horn; every member of S is dual Horn; every member of S is affine. Otherwise, we say that S is *non-Schaefer*. ∎

There are efficient criteria to determine whether a logical relation is 2CNF, Horn, dual Horn, or affine. In fact, a set of such criteria was already provided by Schaefer [Sch78]; moreover, even simpler criteria for a relation to be Horn or dual Horn were given by Dechter and Pearl [DP92]. Each of these criteria involves a *closure property* of the logical relations at hand under a certain function. Specifically, a relation R is 2CNF if and only if for all $t_1, t_2, t_3 \in R$, we have that $(t_1 \vee t_2) \wedge (t_2 \vee t_3) \wedge (t_1 \vee t_3) \in R$, where the operators $\vee$ and $\wedge$ are applied coordinate-wise to bit tuples. R is Horn (respectively, dual Horn) if and only if for all $t_1, t_2 \in R$, we have that $t_1 \wedge t_2 \in R$ (respectively, $t_1 \vee t_2 \in R$). Finally, R is affine if and only if for all $t_1, t_2, t_3 \in R$, we have that $t_1 \oplus t_2 \oplus t_3 \in R$.

If S is a 0-valid or a 1-valid set of logical relations, then $\textsc{Sat}(S)$ is a trivial decision problem (the answer is always "yes"). If S is an affine set of logical relations, then $\textsc{Sat}(S)$ can be solved in polynomial time using Gaussian elimination. Moreover, there are well-known polynomial-time algorithms for the satisfiability problem for the class of all 2CNF formulas (2-$\textsc{Sat}$), the class of all Horn formulas, and the class of all dual Horn formulas. Schaefer's seminal discovery was that the above six cases are the *only* tractable cases of $\textsc{Sat}(S)$; furthermore, the last four are the *only* tractable cases of $\textsc{Sat}_{\mathrm{C}}(S)$.

Theorem 2.2: [Dichotomy Theorems, [Sch78]]
Let S be a finite set of logical relations.

If S is 0-valid or 1-valid or Schaefer, then $\textsc{Sat}(S)$ is solvable in polynomial time; otherwise, it is NP*-complete.*

If S is Schaefer, then $\textsc{Sat}_{\mathrm{C}}(S)$ is solvable in polynomial time; otherwise, it is NP*-complete.*

Theorem 2.2 immediately implies that $\textsc{Positive-1-In-3-Sat}$ is NP-complete, since this is the same problem as $\textsc{Sat}(R_{1/3})$, and $R_{1/3}$ is neither 0-valid, nor 1-valid, nor Schaefer, as can be seen by applying the aforementioned closure properties.

To obtain the above dichotomy theorems, Schaefer had to first establish a result asserting that every non-Schaefer set S has extremely high expressive power, in the sense that every logical relation can be defined from an $\mathcal{F}_{\mathrm{C}}(S)$-formula using existential quantification.

Theorem 2.3: [Expressibility Theorem, [Sch78]]
Let S be a finite set of logical relations. If S is non-Schaefer, then for every k-ary logical relation R there is an $\mathcal{F}_{\mathrm{C}}(S)$-formula $\varphi(x_1, \ldots, x_k, z_1, \ldots, z_m)$ such that R coincides with the set of all truth assignments to the variables $x_1, \ldots, x_k$ that satisfy the formula $(\exists \bar{z})\varphi(\bar{x}, \bar{z})$.

3 Propositional Circumscription

In circumscription, properties are specified in some logical formalism, a natural partial order between models of each formula is considered, and the focus is on models that are minimal with respect to this partial order. Minimal models are preferred because they have as few "exceptions" as possible and thus embody common sense. In propositional circumscription, properties are specified using propositional formulas and the focus is on models that are minimal with respect to the coordinate-wise partial order between truth assignments, as defined below.

Let $k \geq 1$ be an integer and let $\alpha = (a_1, \ldots, a_k)$, $\beta = (b_1, \ldots, b_k)$ be two k-tuples in $\{0,1\}^k$. We write $\beta \leq \alpha$ to denote that, for every $i \leq k$, we have that $b_i \leq a_i$ (as usual, $0 \leq 1$). Also, $\beta < \alpha$ means that $\beta \leq \alpha$ and $\beta \neq \alpha$. If φ is a propositional formula and α is a truth assignment to the variables of φ, then we say that α is a *minimal model* of φ if α satisfies φ and no truth assignment $\beta < \alpha$ satisfies φ.

Let φ and ψ be two propositional formulas in CNF. We say that *ψ can be inferred from φ under propositional circumscription*, and write $\varphi \models_{\mathrm{CIRC}} \psi$, if ψ is true in every minimal model of φ. Clearly, if ψ is a conjunction $\bigwedge_{i=1}^{m} c_i$ of clauses c_i, then $\varphi \models_{\mathrm{CIRC}} \psi$ if and only if $\varphi \models_{\mathrm{CIRC}} c_i$, for every $i \leq m$. Thus, the inference problem for propositional circumscription can be stated as follows: given a propositional formula φ in CNF and a clause ψ, does $\varphi \models_{\mathrm{CIRC}} \psi$? Since testing a truth assignment for minimality is in coNP, it follows that the inference problem for propositional circumscription is in Π_2^{P}. As mentioned earlier, in [EG93] this problem was shown to be Π_2^{P}-complete, even when φ is a 3CNF-formula and ψ is just a negative literal $\neg u$. Our goal is to investigate the complexity of the inference problem for propositional circumscription in the context of Schaefer's framework. More precisely, each set S of logical relations gives rise to the following decision problem $\textsc{Inf-Circ}(\mathcal{F}_{\mathrm{C}}(S))$: given an $\mathcal{F}_{\mathrm{C}}(S)$-formula φ and a clause ψ, does $\varphi \models_{\mathrm{CIRC}} \psi$? The next proposition asserts that each of these decision problems is equivalent to a special case of it.

Proposition 3.1 : *For every set S of logical relations, $\textsc{Inf-Circ}(\mathcal{F}_{\mathrm{C}}(S))$ is equivalent to the following decision problem: given an $\mathcal{F}_{\mathrm{C}}(S)$-formula φ and a negative clause $(\neg u_1 \vee \cdots \vee \neg u_n)$, does $\varphi \models_{\mathrm{CIRC}} (\neg u_1 \vee \cdots \vee \neg u_n)$?*

Proof: Given an $\mathcal{F}_{\mathrm{C}}$-formula φ and a clause $(x_1 \vee \cdots \vee x_m \vee \neg u_1 \vee \cdots \vee \neg u_n)$, let φ' be the $\mathcal{F}_{\mathrm{C}}$-formula obtained from φ by replacing each occurrence of x_i, $1 \leq i \leq m$, by $\mathbf{0}$. It is easy to verify that $\varphi \models_{\mathrm{CIRC}} (x_1 \vee \cdots \vee x_m \vee \neg u_1 \vee \cdots \vee \neg y_n)$ if and only if $\varphi' \models_{\mathrm{CIRC}} (\neg u_1 \vee \cdots \vee u_n)$. ∎

Consider the following restricted case of $\textsc{Inf-Circ}(\mathcal{F}_{\mathrm{C}}(S))$: given an $\mathcal{F}_{\mathrm{C}}(S)$-formula φ and a positive clause $(x_1 \vee \cdots \vee x_m)$, does $\varphi \models_{\mathrm{CIRC}} (x_1 \vee \cdots \vee x_m)$? This problem is in coNP, because it is easy to check that $\varphi \models_{\mathrm{CIRC}} (x_1 \vee \cdots \vee x_m)$ if and only if $\varphi \models (x_1 \vee \cdots \vee x_m)$. Thus, the inference of clauses with negative literals is essential in establishing that certain $\textsc{Inf-Circ}(\mathcal{F}_{\mathrm{C}}(S))$ problems are Π_2^{P}-complete.

We are now ready to state the main results of this paper. These results classify the complexity of all $\textsc{Inf-Circ}(\mathcal{F}_{\mathrm{C}}(S))$ problems and, in particular, give efficiently checkable criteria that characterize when $\textsc{Inf-Circ}(\mathcal{F}_{\mathrm{C}}(S))$ is a Π_2^{P}-complete problem. As mentioned in Section 1, we first establish a dichotomy theorem for $\textsc{Inf-Circ}(\mathcal{F}_{\mathrm{C}}(S))$, where S is assumed to be a 1-valid set of logical relations, i.e., every relation in S contains the all-ones tuple $(1, 1, \ldots, 1)$.

Theorem 3.2: *Let S be a 1-valid set of logical relations.*

If S is Schaefer, then $\textsc{Inf-Circ}(\mathcal{F}_{\mathrm{C}}(S))$ is in coNP*; otherwise, it is Π_2^{P}-complete. Actually, if S is non-Schaefer, then even the following special case of $\textsc{Inf-Circ}(\mathcal{F}_{\mathrm{C}}(S))$ is Π_2^{P}-complete: given an $\mathcal{F}_{\mathrm{C}}(S)$-formula φ and a negative literal $\neg u$, does $\varphi \models_{\mathrm{CIRC}} \neg u$?*

Moreover, there is a polynomial-time algorithm to decide whether, given a finite 1-valid set of logical relations, $\textsc{Inf-Circ}(\mathcal{F}_{\mathrm{C}}(S))$ is in coNP *or Π_2^{P}-complete.*

An outline of the proof of Theorem 3.2 is presented in Section 4. The following examples illustrate the preceding Theorem 3.2 and provide new instances of restricted cases of the inference problem for propositional circumscription having the same inherent complexity as the general case.

Example 3.3: Consider the ternary logical relation $K = \{(1,1,1), (0,1,0), (0,0,1)\}$. Using the closure properties that characterize when a logical relation is 2CNF, Horn, dual Horn, or affine, it is easy to see that K is none of the above. For instance, K is not Horn because $(0,1,0) \wedge (0,0,1) = (0,0,0) \notin K$. Consequently, Theorem 3.2 implies that $\textsc{Inf-Circ}(\mathcal{F}_{\mathrm{C}}(\{K\}))$ is Π_2^{P}-complete. ∎

Example 3.4: Consider the 1-valid set $S = \{R_0, R_1, R_2\}$, where $R_0 = \{0,1\}^3 - \{(0,0,0)\}$ (expressing the clause $(x \vee y \vee z)$), $R_1 = \{0,1\}^3 - \{(1,0,0)\}$ (expressing the clause $(\neg x \vee y \vee z)$), $R_2 = \{0,1\}^3 - \{(1,1,0)\}$ (expressing the clause $(\neg x \vee \neg y \vee z)$). Using the closure properties, it is easy to verify that R_1 is neither 2CNF, nor Horn, nor affine, and that R_2 is not dual Horn. Consequently, Theorem 3.2 implies that $\textsc{Inf-Circ}(\mathcal{F}_{\mathrm{C}}(S))$ is Π_2^{P}-complete. ∎

As mentioned in Section 1, Theorem 3.2 can be used as stepping stone to obtain a dichotomy theorem for the family of all $\textsc{Inf-Circ}(\mathcal{F}_{\mathrm{C}}(S))$ problems, where S is an arbitrary set of logical relations. To this effect, we use the following crucial concept, which was first introduced in [KK01].

Definition 3.5: Let R be a k-ary logical relation. We say that a logical relation T is a 0-*section* of R if either T is the relation R itself or T can be defined from the formula $R(x_1, \ldots, x_k)$ by replacing at least one, but not all, of the variables $x_1, \ldots, x_k$ by 0. ∎

To illustrate this concept, consider the logical relation $R_{1/3} = \{(1,0,0), (0,1,0), (0,0,1)\}$. Then the logical relation $\{1\}$ is a 0-section of $R_{1/3}$, since it is definable by $R_{1/3}(x_1, 0, 0)$. In fact, it is easy to see that $\{(1)\}$ is the *only* logical relation that is both 1-valid and a 0-section of $R_{1/3}$.

Theorem 3.6: *Let S be a set of logical relations and let S^* be the set of all logical relations P such that P is both 1-valid and a 0-section of some relation in S.*

If S^* is Schaefer, then $\textsc{Inf-Circ}(\mathcal{F}_{\mathrm{C}}(S))$ is in coNP*; otherwise, it is Π_2^{P}-complete. Actually, if S^* is non-Schaefer, then even the following special case of $\textsc{Inf-Circ}(\mathcal{F}_{\mathrm{C}}(S))$ is Π_2^{P}-complete: given an $\mathcal{F}_{\mathrm{C}}(S)$-formula φ and a negative literal $\neg u$, does $\varphi \models_{\mathrm{CIRC}} \neg u$?*

Moreover, there is a polynomial-time algorithm to decide whether, given a finite set S of logical relations, $\textsc{Inf-Circ}(\mathcal{F}_{\mathrm{C}}(S))$ is in coNP *or Π_2^{P}-complete.*

The proof of Theorem 3.6 will be given in the full paper. We now present several different examples that illustrate the power of Theorem 3.6. The first shows how the main result in [EG93] can be easily derived from Theorem 3.6.

Example 3.7: Recall that 3-$\textsc{Sat}$ coincides with $\textsc{Sat}(S)$, where $S = \{R_0, R_1, R_2, R_3\}$ and $R_0 = \{0,1\}^3 - \{(0,0,0)\}$ (expressing the clause $(x \vee y \vee z)$), $R_1 = \{0,1\}^3 - \{(1,0,0)\}$ (expressing the clause $(\neg x \vee y \vee z)$), $R_2 = \{0,1\}^3 - \{(1,1,0)\}$ (expressing the clause $(\neg x \vee \neg y \vee z)$), and $R_3 = \{0,1\}^3 - \{(1,1,1)\}$ (expressing the clause $(\neg x \vee \neg y \vee \neg z)$).

Since the logical relations R_0, R_1, R_2 are 1-valid, they are members of S^*. It follows that S^* is not Schaefer, since R_1 is not 2CNF or Horn or affine, and R_2 is not dual Horn. Theorem 3.6 immediately implies that $\textsc{Inf-Circ}(\mathcal{F}_{\mathrm{C}}(S))$ (i.e., $\textsc{Inf-Circ}(\mathrm{3CNF})$) is Π_2^{P}-complete. ∎

Example 3.8: Consider the set $S = \{R_0, R_3\}$, where R_0 and R_3 are as in the preceding Example 3.7. In this case, SAT(S) is the problem MONOTONE 3-SAT, that is to say, the restriction of 3-SAT to 3CNF-formulas in which every clause is either the disjunction of positive literals or the disjunction of negative literals. It is well known that this problem is NP-complete (this can also be derived from Schaefer's Dichotomy Theorem 2.2). It is not hard to verify that every relation in S^* is dual Horn (for instance, S^* contains R_0, which is dual Horn). Consequently, Theorem 3.6 implies that INF-CIRC($\mathcal{F}_{\mathrm{C}}(S)$) is in coNP. ∎

The preceding example reveals that the boundary in the dichotomy for the inference problem in classical propositional logic is different than that in the dichotomy for the inference problem in propositional circumscription. Several other instances of this phenomenon are provided by the final example of this section.

Example 3.9: If m and n are two positive integers with $m < n$, then $R_{m/n}$ is the n-ary logical relation consisting of all n-tuples that have m ones and $n - m$ zeros. It is easy to see that $R_{m/n}$ is not Schaefer. Consequently, if S is a set of logical relations each of which is of the form $R_{m/n}$ for some m and n with $m < n$, then SAT(S) is NP-complete. On the other hand, S^* is easily seen to be Horn (and, hence, Schaefer), since every relation P in S^* is a singleton $P = \{(1, \ldots, 1)\}$ consisting of the m-ary all-ones tuple for some m. Consequently, Theorem 3.6 implies that INF-CIRC($\mathcal{F}_{\mathrm{C}}(S)$) is in coNP.

This family of examples contains POSITIVE-1-IN-3-SAT as the special case where $S = \{R_{1/3}\}$. ∎

4 Outline of Proof of Theorem 3.2

In this section, we present an outline of the dichotomy theorem for INF-CIRC($\mathcal{F}(S)$), where S is a 1-valid set of logical relations. Due to space limitations, we have to confine ourselves to stating the main technical steps and to making a few high-level comments.

Assume first that S is Schaefer. In this case, is easy to see that there is a polynomial-time algorithm to decide whether a given model of an $\mathcal{F}_{\mathrm{C}}(S)$-formula is minimal. From this fact, it follows immediately that if S is Schaefer, then INF-CIRC($\mathcal{F}_{\mathrm{C}}(S)$) is in coNP.

Towards the Π_2^{P}-hardness result, assume that S is not Schaefer. Using Schaefer's Expessibility Theorem 2.3, the following decision problem can be shown to be Π_2^{P}-complete: Given a $\mathcal{F}(S)$-formula $\varphi(\bar{x}, \bar{y}, w_0, w_1)$, decide whether the sentence $\forall\bar{x}\exists\bar{y}\varphi(\bar{x}, \bar{y}, \mathbf{0}/w_0, \mathbf{1}/w_1)$ is true. Our goal is to show that this problem has a polynomial-time reduction to INF-CIRC($\mathcal{F}(S)$). One of the key steps in the reduction is the following lemma, which was inspired from a result in [EG93]. A proof can be found in the Appendix.

Lemma 4.1: *Let S be 1-valid set and let $\varphi(\bar{x}, \bar{y}, w_0, w_1)$ be an $\mathcal{F}(S)$-formula, where $\bar{x} = (x_1, \ldots, x_n)$, $\bar{y} = (y_1, \ldots, y_m)$, w_0 and w_1 is the list of its variables. Let u, $\bar{x}' = (x'_1, \ldots, x'_n)$ and $\bar{z} = (z_1, \ldots, z_n)$ be new variables, and let $\chi(u, \bar{x}, \bar{z}, \bar{x}', \bar{y})$ be the following formula*

$$\varphi(\bar{x}', \bar{y}, u/w_0, \mathbf{1}/w_1) \wedge \left(\bigwedge_{i=1}^{n} (x_i \not\equiv z_i)\right) \wedge$$

$$\left(\bigwedge_{j=1}^{m} (u \rightarrow y_j)\right) \wedge \left(\bigwedge_{i=1}^{n} (x'_i \equiv (u \vee x_i))\right).$$

Then the formula $\forall\bar{x}\exists\bar{y}\varphi(\bar{x}, \bar{y}, \mathbf{0}/w_0, \mathbf{1}/w_1)$ is true if and only if $\chi(u, \bar{x}, \bar{z}, \bar{x}', \bar{y}) \models_{\mathrm{CIRC}} \neg u$.

Although φ is an $\mathcal{F}_{\mathrm{C}}(S)$-formula, the formula χ in the preceding lemma is *not* an $\mathcal{F}_{\mathrm{C}}(S)$-formula, because it contains elementary connectives, such as $\equiv$, $\rightarrow$, and $\vee$. So, the task now is to construct an $\mathcal{F}_{\mathrm{C}}(S)$-formula θ in polynomial time such that $\chi \models_{\mathrm{CIRC}} \neg u$ if and only if $\theta \models_{\mathrm{CIRC}} \neg u$. It is now natural to apply Schaefer's Expressibility Theorem 2.3 again and express each of the above elementary connectives using an $\exists\mathcal{F}_{\mathrm{C}}(S)$-formula, i.e., a formula of the form $\exists\bar{w}\zeta$, where ζ is an $\mathcal{F}_{\mathrm{C}}(S)$-formula. After these steps are completed, we obtain an $\exists\mathcal{F}_{\mathrm{C}}(S)$-formula $\exists\bar{v}\chi'$ with the same free variables as χ such that $\chi \models_{\mathrm{CIRC}} \neg u$ if and only if $\exists\bar{v}\chi' \models_{\mathrm{CIRC}} \neg u$. At this point, one may be tempted to simply drop the existential quantifiers $\exists\bar{v}$, focus on the $\exists\mathcal{F}_{\mathrm{C}}(S)$-formula χ', and claim that $\chi \models_{\mathrm{CIRC}} \neg u$ if and only if $\chi' \models_{\mathrm{CIRC}} \neg u$. The flaw in this argument is that Schaefer's Expressibility Theorem 2.3 gives no explicit information about the possible values of the existential quantifiers in $\exists\mathcal{F}_{\mathrm{C}}(S)$-formulas expressing logical relations. As a result, the witnesses to the variables $\bar{v}$ in the existential quantifiers $\exists\bar{v}$ may not give rise to minimal satisfying truth assignments of χ', hence the claimed equivalence may fail.

To bypass this serious obstacle, we must give up applying Schaefer's Expressibility Theorem 2.3 and instead have to use certain expressibility lemmas to the effect that all necessary elementary connectives are definable by $\exists\mathcal{F}_{\mathrm{C}}(S)$-formulas with explicit information about the witnesses to the existential quantifiers. The first of these lemmas, due to Creignou and Hébrard [CH97], concerns the definability of the connectives $\rightarrow$ and $\vee$; it also brings out the importance of the logical relation K introduced in Example 3.3. In what follows, $\mathcal{F}_1(S)$ denotes the class of all formulas obtained from $\mathcal{F}(S)$-formulas by substituting some variables by the constant $\mathbf{1}$.

Lemma 4.2: (Creignou and Hébrard [CH97]) *Let S be a 1-valid, non-Schaefer set of logical relations. Then at least one of the following two statements is true.*

1. *There exists an $\mathcal{F}_1(S)$-formula $\varepsilon(x, y)$ with the property that $(x \rightarrow y) \equiv \varepsilon(x, y)$.*

2. *The logical relation* $K = \{(1,1,1),(0,1,0),(0,0,1)\}$ *is in* $\mathcal{F}_1(S)$, *i.e., there exists an* $\mathcal{F}_1(S)$*-formula* $\kappa(x,y,z)$ *which is satisfied* only *by the three truth assignments* $(1,1,1),(0,1,0)$ *and* $(0,0,1)$. *Therefore:*

   (i) $(x \rightarrow y) \equiv (\exists z)\kappa(x,y,z)$; *moreover,* $(\exists z)\kappa(x,y,z)$ *has the additional property that* 1 *is the only witness for the variable* z *under the truth assignment* $(1,1)$ *to the variables* (x,y).

   (ii) $(x \vee y) \equiv (\exists z)\kappa(z,x,y)$; *moreover,* $(\exists z)\kappa(z,x,y)$ *has the additional property that* 1 *is the only witness for the variable* z *under the truth assignment* $(1,1)$ *to the variables* (x,y).

The second expressibility lemma concerns the definability of the connective $\equiv$.

Lemma 4.3: *Let* S *be a* 1*-valid, non-Schaefer set of logical relations. Then there exists a three-variable* $\mathcal{F}_1(S)$*-formula* $\kappa'(x,y,z)$ *that is satisfied by the truth assignments* $(1,1,1)$, $(1,0,0)$ *and* $(0,0,1)$ *but is not satisfied by the truth assignment* $(1,0,1)$ *(no information about the remaining four possible assignments is required). Moreover, if we set* $\lambda(x',u,z,z')$ *to be the formula*

$$(u \rightarrow x') \wedge (x' \vee z) \wedge (z \rightarrow z') \wedge (u \rightarrow z') \wedge \kappa'(x',u,z'),$$

we have the following properties:

(i) the formula $x' \equiv (u \vee \neg z)$ *is logically equivalent to the formula* $(\exists z')\lambda(x',u,z,z')$;

(ii) the only witnesses z' *for each of the four assignments* $(x'=1,u=1,z=0),(x'=1,u=0,z=0),(x'=1,u=1,z=1)$ *and* $(x'=0,u=0,z=1)$ *that satisfy the formula* $(\exists z')\lambda(x',u,z,z')$ *are* $z'=1, z'=0, z'=1$ *and* $z'=1$, *respectively.*

The proof of Lemma 4.3 can be found in the Appendix, which also contains a self-contained proof of Lemma 4.2, since that proof is used in the proof of Lemma 4.3.

We are now ready to return to the proof of Theorem 3.2. As stated earlier, our goal is to show that the following problem has a polynomial-time reduction to INF-CIRC$(()\mathcal{F}_{\mathrm{C}}(S))$: given a $\mathcal{F}(S)$-formula $\varphi(\bar{x},\bar{y},w_0,w_1)$, decide whether the sentence $\forall\bar{x}\exists\bar{y}\varphi(\bar{x},\bar{y},\mathbf{0}/w_0,\mathbf{1}/w_1)$ is true. Towards this goal, we start with the formula χ described in Lemma 4.1 and then adjust χ in six successive steps $l = 1,\ldots,6$ (enumerated below). At the last step, we will have constructed an $\mathcal{F}_{\mathrm{C}}(S)$-formula for which the desired reduction holds. More formally, at each step $l = 1,\ldots,6$, we will construct a formula χ_l such that for all $l = 0,\ldots,5$ (assuming that χ_0 is χ), the set of free variables of χ_l is going to be a subset (not necessarily proper) of χ_{l+1} and, in addition, the formulas χ_l will satisfy the following three requirements:

R1: Every truth assignment that satisfies χ_l can be extended to a truth assignment that satisfies χ_{l+1}.

R2: The restriction of every truth assignment that satisfies χ_{l+1} to the variables of χ_l also satisfies χ_l.

R3: Let α and α' be two satisfying truth assignments of χ_l such that $\alpha(u) = 1$ and $\alpha' \leq \alpha$. If β is an extension of α to a satisfying truth assignment of χ_{l+1}, then there is an extension β' of α' to a satisfying truth assignment of χ_{l+1} such that $\beta' \leq \beta$.

It is easy to see that once we prove the above three requirements, then for each $l \geq 0$, χ_l has a minimal satisfying truth assignment with $u = 1$ if and only if χ_{l+1} does. From Lemma 4.1 and the fact that the formula constructed at the last step will be in $\mathcal{F}_{\mathrm{C}}(S)$, it follows that the reduction will be complete.

Notice first that if χ_l and χ_{l+1} have the same set of free variables, then the above three requirements are equivalent to asserting that χ_l and χ_{l+1} are logically equivalent.

Step 1: In χ, replace each connective $x'_i \equiv (u \vee x_i)$, for $i = 1,\ldots,n$, with $x'_i \equiv (u \vee \neg z_i)$. The formula χ_1 has the same variables as χ and it is equivalent to χ, since the conjunct $\bigwedge_{i=1}^n (x_i \not\equiv z_i)$ appears in both χ and χ_1 Therefore the requirements R1–R3 are satisfied.

Step 2: In χ_1, replace each connective $x'_i \equiv (u \vee \neg z_i)$, for $i = 1,\ldots,n$, by $\lambda(x'_i,u,z_i,z'_i)$, where the z'_i, for $i = 1,\ldots,n$, are new variables and λ is the formula described in Lemma 4.3. Because of the equivalence of $x'_i \equiv (u \vee \neg z_i)$ with $(\exists z'_i)\lambda(x'_i,u,z_i,z'_i)$, we can immediately conclude that the requirements R1 and R2 are satisfied. To prove requirement R3, observe that because only the variables x'_i,u,z_i, for $i = 1,\ldots,n$, are involved in the connectives that are replaced at the current step, and because we have associated a different witness z'_i for each triple of variables x'_i,u,z_i, we can restrict our attention to assignments to the three variables x'_i,u and z_i only (for an arbitrary but fixed i). Suppose that α and α' are two assignments to x'_i,u and z_i such that α' is less than or equal to α and $u = 1$ in α. Then first observe that because of the conjunct $x'_i \equiv (u \vee \neg z_i)$, $x'_i = 1$ in α. Also observe that because of the conjunct $x_i \not\equiv z_i$, the values of z_i in α and α' are equal (recall from the proof of the Key Lemma 4.1 that we express this fact by saying that the value of z_i, as well as x_i, remain "fixed"). The proof of this step can then be completed by distinguishing two cases according to the common value of z_i in α and α'. The details will appear in the full paper.

Step 3: In χ_2, replace each connective $x'_i \vee z_i$ (that appears as part of the formula $\lambda(x'_i,u,z_i,z'_i)$) by $x_i \rightarrow x'_i$. The satisfaction of the requirments R1–R3 is proved exactly as in Step 1.

Observe that, apart from the conjunct $\bigwedge_{i=1}^n (x_i \not\equiv z_i)$, the only logical connectives that have not yet been replaced by an $\mathcal{F}_{\mathrm{C}}(S)$-formula are connectives of the form $x \rightarrow y$

(x and y are used as generic names of variables), where x is either u or x_i or z_i for some i. In the next two steps, we deal with these connectives. Notice first that if the relation $K = \{(1,1,1),(0,1,0,(0,01)\}$ is not in $\mathcal{F}_1(S)$, then we are in Case 1 of Lemma 4.2, therefore there is an $\mathcal{F}_1(S)$ formula $\varepsilon(x,y)$ equivalent to $x \rightarrow y$. In this case, in one step that subsumes the following two steps, we just replace every occurrence of $x \rightarrow y$ with $\varepsilon(x,y)$. So in the next two steps, we assume that the relation K is in $\mathcal{F}_1(S)$, and therefore we are in Case 2 of Lemma 4.2.

Step 4: In χ_3, replace each connective $u \rightarrow x$ (x is again a generic name for variables) with $\kappa(u,x,x')$, where x' is a new variable distinct for each x and κ is the formula described in Case 2 of Lemma 4.2. The validity of the requirements R1 and R2 is immediate. As for requirement R3, restrict attention to the variables u and x, for an arbitrary but fixed variable x. The validity of R3 then follows from the witness property (i) established in Lemma 4.2.

Step 5: Notice first that we cannot imitate Step 4 and replace the connectives of the form $x_i \rightarrow x$ with $\kappa(x_i,x,x')$, since in two models α and α' of $x_i \rightarrow x$ such that α' is less than or equal to α, the value of x_i remains fixed, while it is the value of x that may drop from 1 in α to 0 in α'. Therefore, the witness property (i) of Lemma 4.2 does not suffice to prove R3 for the case when $x_i = 0$. Instead, we first substitute $x_i \rightarrow x$ with $z_i \vee x$ and then substitute the latter with $\kappa(x', z_i, x)$. If we use the witness property (ii) in Lemma 4.2 for the connective $z_i \vee x$, everything goes through, for both possibilities $z_i = 1$ and $z_i = 0$, as it can be easily seen. We deal similarly with the connectives of the form $z_i \rightarrow x$.

Step 6: By Schaefer's Expressibility Theorem 2.3, there is an $\mathcal{F}_1(S)$ formula, say $\zeta(x,y,t_1,\ldots,t_s,w_0)$, such that for each $i = 1,\ldots,n$, the connective $x_i \not\equiv z_i$ is logically equivalent to $(\exists\bar{t})\zeta(x_i/x, z_i/y, \bar{t}, 0/w_0)$. To construct χ_6, replace in χ_5 the connectives $x_i \not\equiv z_i$ with $\zeta(x_i/x, z_i/y, x''_{i,1}/t_1, \ldots, x''_{i,s}/t_s, 0/w_0)$, where $x''_{i,r}$ for $i = 1,\ldots,n$ and $r = 1,\ldots,s$ are new variables. It is not hard to see that requirements R1–R3 can be proved in this case with no special properties for the witnesses. Notice that χ_6 is in $\mathcal{F}_{\mathrm{C}}(S)$ (and that the constant 0 was only used in the last step).

This concludes the outline of the proof of Theorem 3.2. ∎

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References

[CH96] N. Creignou and M. Hermann. Complexity of generalized satisfiability counting problems. *Information and Computation*, 125(1):1–12, 1996.

[CH97] N. Creignou and J.-J. Hébrard. On generating all solutions of generalized satisfiability problems. *Theoretical Informatics and Applications*, 31(6):499–511, 1997.

[CL94] M. Cadoli and M. Lenzerini. The complexity of closed world reasoning and circumscription. *Journal of Information and System Sciences*, pages 255–301, 1994.

[Cre95] N. Creignou. A dichotomy theorem for maximum generalized satisfiability problems. *Journal of Computer and System Science*, 51(3):511–522, 1995.

[DP92] R. Dechter and J. Pearl. Structure identification in relational data. *Artificial Intelligence*, 48:237–270, 1992.

[EG93] Th. Eiter and G. Gottlob. Propositional circumscription and extended closed-world reasoning are Π_2^{P}-complete. *Theoretical Computer Science*, 114:231–245, 1993.

[GJ79] M. R. Garey and D. S. Johnson. *Computers and Intractability - A Guide to the Theory of NP-Completeness*. W. H. Freeman and Co., 1979.

[GPP89] M. Gelfond, H. Przymusinska, and T. Przymusinksi. On the relationship between circumscription and negation as failure. *Artificial Intelligence*, 38:273–287, 1989.

[KK01] L.M. Kirousis and Ph.G. Kolaitis. The complexity of minimal satisfiability problems. In *Proc. of the 18th Annual Symposium on Theoretical Aspects of Computer Science - STACS 2001*, volume 2010 of *LNCS*, Springer 2001.

[KS98] D. Kavvadias and M. Sideri. The inverse satisfiability problem. *SIAM Journal of Computing*, 28(1):152–163, 1998.

[KSW97] S. Khanna, M. Sudan, and D.P. Williamson. A complete classification of the approximability of maximization problems derived from Boolean constraint satisfaction. In *Proc. of the 29th Annual ACM Symposium on Theory of Computing*, pages 11–20, 1997.

[Lad75] R. E. Ladner. On the structure of polynomial time reducibility. *Journal of the Association for Computing Machinery*, 22(1):155–171, 1975.

[McC80] J. McCarthy. Circumscription - a form of non-monotonic reasoning. *Artificial Intelligence*, 13:27–39, 1980.

[Pap94] C. H. Papadimitriou. *Computational Complexity*. Addison-Wesley Publishing Company, 1994.

[RV00] S. Reith and H. Vollmer. Oprimal satisfiability of propositional calculi and constraint satisfaction problems. In *25th Int. Symp. on Mathematical Foundations of Computer Science - MFCS 2000*, volume 1893 of *LNCS*, pages 640–649. Springer, 2000.

[Sch78] T.J. Schaefer. The complexity of satisfiability problems. In *Proc. 10th ACM Symp. on Theory of Computing*, pages 216–226, 1978.

Appendix: Proof of Lemmas 4.1, 4.2, and 4.3

Lemma 4.1: Let S be 1-valid set and let $\varphi(\bar{x}, \bar{y}, w_0, w_1)$ be an $\mathcal{F}(S)$-formula, where $\bar{x} = (x_1, \ldots, x_n)$, $\bar{y} = (y_1, \ldots, y_m)$, w_0 and w_1 is the list of its variables. Let u, $\bar{x}' = (x'_1, \ldots, x'_n)$ and $\bar{z} = (z_1, \ldots, z_n)$ be new variables, and let $\chi(u, \bar{x}, \bar{z}, \bar{x}', \bar{y})$ be the following formula

$$\varphi(\bar{x}', \bar{y}, u/w_0, \mathbf{1}/w_1) \wedge \left(\bigwedge_{i=1}^{n} (x_i \not\equiv z_i) \right) \wedge$$

$$\left(\bigwedge_{j=1}^{m} (u \rightarrow y_j) \right) \wedge \left(\bigwedge_{i=1}^{n} (x'_i \equiv (u \vee x_i)) \right).$$

Then the formula $\forall \bar{x} \exists \bar{y} \varphi(\bar{x}, \bar{y}, \mathbf{0}/w_0, \mathbf{1}/w_1)$ is true if and only if $\chi(u, \bar{x}, \bar{z}, \bar{x}', \bar{y}) \models_{\text{CIRC}} \neg u$.

Proof: For the if part, consider an assignment α to the variables $\bar{x}$ that satisfies the formula $\forall \bar{y} \neg \varphi(\bar{x}, \bar{y}, \mathbf{0}/w_0, \mathbf{1}/w_1)$. Extend α to an assignment β of all variables of the formula χ by letting $u = 1$, $x'_i = 1$ for $i = 1, \ldots, n$, $y_j = 1$ for $j = 1, \ldots, m$, and by giving to each z_i, for $i = 1, \ldots, n$, the opposite value of x_i. Because φ is 1-valid, it is easy to see that β satisfies χ. We will show that β is actually a minimal satisfying assignment of χ. First observe that the conjuncts $\bigwedge_{i=1}^{n} (x_i \not\equiv z_i)$ ensure that none of the variables $\bar{x}$ or $\bar{z}$ can get a different value at a satisfying assignment of χ strictly smaller than β (we express this fact by saying that the values of $\bar{x}$ and $\bar{z}$ are fixed). Also, the conjuncts $\bigwedge_{j=1}^{m} (u \rightarrow y_j)$ and $\bigwedge_{i=1}^{n} (x'_i \equiv (u \vee x_i))$ ensure that the values of $\bar{y}$ and $\bar{x}'$ are bound to be 1 at any assignment satisfying χ and with $u = 1$. All we have to prove is that u cannot get the value 0 at a satisfying assignment of χ smaller than β. Assume it did and let $\gamma < \beta$ be be a satisfying assignment of χ with $u = 0$. Then, observe that in γ, because of the conjunct $\bigwedge_{i=1}^{n} (x'_i \equiv (u \vee x_i))$, the values of $\bar{x}'$ would be equal to the corresponding values of $\bar{x}$. Therefore, because of the first conjunct of χ, and because $u = 0$ in γ, the values of $\bar{x}$ and $\bar{y}$ in γ would satisfy $\varphi(\bar{x}, \bar{y}, \mathbf{0}/w_0, \mathbf{1}/w_1)$. Now observe that γ and β coincide on $\bar{x}$, because the value of $\bar{x}$ is "fixed". Therefore γ and α also coincide on $\bar{x}$, since by construction β extends α. This is a contradiction, because we assumed that α satisfies $\forall \bar{y} \neg \varphi(\bar{x}, \bar{y}, \mathbf{0}/w_0, \mathbf{1}/w_1)$.

To prove the converse, consider a minimal assignment α of χ with $u = 1$ and also consider the assignment β induced by α on $\bar{x}$. We claim that β satisfies $\forall \bar{y} \neg \varphi(\bar{x}, \bar{y}, \mathbf{0}/w_0, \mathbf{1}/w_1)$. If not, then there is an assignment of values to $\bar{y}$ which combined with β forms an assignment γ that satisfies $\varphi(\bar{x}, \bar{y}, \mathbf{0}/w_0, \mathbf{1}/w_1)$. Extend γ to an assignment δ of all variables of χ by setting $u = 0$, $x'_i = x_i$ for $i = 1, \ldots, n$, and by giving to each z_i for $i = 1, \ldots, n$ the opposite value of x_i. It is easy to see that δ satisfies χ and is strictly smaller than α, which is a contradiction. ∎

Lemma 4.2: (Creignou and Hébrard [CH97]) *Let S be a* 1*-valid, non-Schaefer set of logical relations. Then at least one of the following two statements is true.*

1. *There exists an $\mathcal{F}_1(S)$-formula $\varepsilon(x, y)$ with the property that $(x \rightarrow y) \equiv \varepsilon(x, y)$.*

2. *The logical relation $K = \{(1,1,1), (0,1,0), (0,0,1)\}$ is in $\mathcal{F}_1(S)$, i.e., there exists an $\mathcal{F}_1(S)$-formula $\kappa(x, y, z)$ which is satisfied* only *by the three truth assignments $(1,1,1), (0,1,0)$ and $(0,0,1)$. Therefore:*

   (i) $(x \rightarrow y) \equiv (\exists z) \kappa(x, y, z)$; moreover, $(\exists z) \kappa(x, y, z)$ has the additional property that 1 is the only witness for the variable z under the truth assignment $(1,1)$ to the variables (x, y).

   (ii) $(x \vee y) \equiv (\exists z) \kappa(z, x, y)$; moreover, $(\exists z) \kappa(z, x, y)$ has the additional property that 1 is the only witness for the variable z under the truth assignment $(1,1)$ to the variables (x, y).

Proof: Since S is a 1-valid, non-Schaefer set of logical relations, it must contain a 1-valid logical relation R that is not affine. As shown in [CH96], there must exist two k-tuples $s, t \in R$ such that $\bar{1} \oplus s \oplus t \notin R$, where $\bar{1}$ is the all-ones k-tuple $(1, \ldots, 1)$ and k is the arity of R. Let $x_1, \ldots, x_k$ be propositional variables and let R' be a relation symbol of arity k that will be interpreted by R. For $(i, j) \in \{0,1\}^2$, let V_{ij} be the set of all variables x_p, $1 \leq p \leq k$, such that the p-th coordinate of the tuple s is equal to i, and the p-th coordinate of the tuple t is equal to j. Let x, y, z, w be four new propositional variables and let $\varphi_1(x, y, z, w)$

be the $\mathcal{F}(S)$-formula $R'(x/V_{00}, y/V_{10}, z/V_{01}, w/V_{11})$ obtained from the formula $R'(x_1, \ldots, x_k)$ by substituting the variable x for all occurrences of the variables in V_{00}, and similarly for the variables y, z, and w. Also let $\varphi_2(x, y, z)$ be the $\mathcal{F}_1(S)$-formula $\varphi_1(x, y, z, \mathbf{1}/w)$. Now observe the following: (1) the truth assignment $(1, 1, 1, 1)$ satisfies $\varphi_1(x, y, z, w)$, since $\bar{1} \in R$; (2) the truth assignment $(0, 1, 0, 1)$ satisfies $\varphi_1(x, y, z, w)$, since $s \in R$; (3) the truth assignment $(0, 0, 1, 1)$ satisfies the $\varphi_1(x, y, z, w)$, since $t \in R$; (4) the truth assignment $(1, 0, 0, 1)$ does not satisfy $\varphi_1(x, y, z, w)$, since $\bar{1} \oplus s \oplus t \notin R$. Therefore, $(1, 1, 1)$, $(0, 1, 0)$ and $(0, 0, 1)$ satisfy $\varphi_2(x, y, z)$, while $(1, 0, 0)$ does not.

We have no information as to whether or not the remaining four assignments $(1, 1, 0), (0, 1, 1), (1, 0, 1)$, $(0, 0, 0)$ satisfy $\varphi_2(x, y, z)$. Thus, we have sixteen possibilities to examine regarding the satisfiability of $\varphi_2(x, y, z)$ by these four truth assignments. We start by branching on the two possibilities for the truth assignment $(0, 0, 0)$:

Case A: $(0, 0, 0)$ satisfies $\varphi_2(x, y, z)$. We distinguish two subcases: Subcase A.1: $(0, 1, 1)$ satisfies $\varphi_2(x, y, z)$. Then set $\varepsilon(x, y) \equiv \varphi_2(x, y, y)$. Subcase A.2: $(0, 1, 1)$ does not satisfy $\varphi_2(x, y, z)$. One more branching: Subcase A.2.1: $(1, 0, 1)$ satisfies $\varphi_2(x, y, z)$.Then set $\varepsilon(x, y) \equiv \varphi_2(y, x, 1)$. Subcase A.2.2: $(1, 0, 1)$ does not satisfy $\varphi_2(x, y, z)$. Then set $\varepsilon(x, y) \equiv \varphi_2(x, y, x)$. This completes the examination of Case A.

Case B: $(0, 0, 0)$ does not satisfy $\varphi_2(x, y, z)$. Consider the following branching: Case B.1: None of the three assignments $(1, 1, 0), (1, 0, 1), (0, 1, 1)$ satisfies $\varphi_2(x, y, z)$. Then $\kappa(x, y, z) \equiv \varphi_2(x, y, z)$. Case B.2: At least one the three assignments $(1, 1, 0), (1, 0, 1), (0, 1, 1)$ satisfies $\varphi_2(x, y, z)$. We make a three-way branching depending on which of these three assignments satisfies $\varphi_2(x, y, z)$. Case B.2.1: $(1, 1, 0)$ satisfies $\varphi_2(x, y, z)$. Then observe that $(x \vee y) \equiv \varphi_2(x, x, y)$. We postpone for a while the continuation of this case where we have already established that $(x \vee y)$ is defined by an $\mathcal{F}_1(S)$-formula. Case B.2.2.: $(1, 0, 1)$ satisfies $\varphi_2(x, y, z)$. Then observe that $(x \vee y) \equiv \varphi_2(x, y, x)$. Again, we postpone the continuation of this case. Case B.2.3: $(0, 1, 1)$ satisfies $\varphi_2(x, y, z)$. Since we have already examined B.2.2, we may assume that $(1, 0, 1)$ does not satisfy $\varphi_2(x, y, z)$. Then set $\varepsilon(x, y) \equiv \varphi_2(x, y, 1)$. At this point all we are left to deal with is the case where $(x \vee y)$ is defined by an $\mathcal{F}_1(S)$-formula. We examine this case below.

Since not every element of S is a dual Horn relation, S must contain a logical relation Q for which there are tuples $s, t \in Q$ such that $s \vee t \notin Q$ (here we use the closure property that characterizes dual Horn relations). By arguments similar to the preceding ones, we can construct an $\mathcal{F}_C(S)$-formula $\psi_2(x, y, z)$ that is satisfied by $(1, 1, 1)$, $(0, 1, 0)$ and $(0, 0, 1)$, but it is not satisfied by $(0, 1, 1)$. Let $\psi_3(x, y, z)$ be the $\mathcal{F}_C(S)$-formula $\psi_2(x, y, z) \wedge (y \vee z)$. Observe that $\psi_3(x, y, z)$ is satisfied by $(1, 1, 1)$, $(0, 1, 0)$ and $(0, 0, 1)$, but it is not satisfied by $(0, 1, 1), (1, 0, 0), (0, 0, 0)$. We are now left with the triples $(1, 1, 0)$ and $(1, 0, 1)$ about which there is no information as to whether they satisfy $\psi_3(x, y, z)$ or not. We consider the following three exhaustive cases:

(1) If $(1, 1, 0)$ satisfies $\psi_3(x, y, z)$, then set $\varepsilon(x, y) \equiv \psi_3(y, 1, x)$; (2) if $(1, 0, 1)$ satisfies $\psi_3(x, y, z)$, then set $\varepsilon(x, y) \equiv \psi_3(y, x, 1)$; (3) if neither $(1, 1, 0)$ nor $(1, 0, 1)$ satisfies $\psi_3(x, y, z)$, then $\kappa(x, y, z) \equiv \psi_3(x, y, z)$. This completes the proof of the Lemma 4.2. ∎

Lemma 4.3: *Let S be a 1-valid, non-Schaefer set of logical relations. Then there exists a three-variable $\mathcal{F}_1(S)$-formula $\kappa'(x, y, z)$ that is satisfied by the truth assignments $(1, 1, 1)$, $(1, 0, 0)$ and $(0, 0, 1)$ but is not satisfied by the truth assignment $(1, 0, 1)$ (no information about the remaining four possible assignments is required). Moreover, if we set $\lambda(x', u, z, z')$ to be the formula*

$$(u \to x') \wedge (x' \vee z) \wedge (z \to z') \wedge (u \to z') \wedge \kappa'(x', u, z'),$$

we have the following properties:

(i) the formula $x' \equiv (u \vee \neg z)$ is logically equivalent to the formula $(\exists z')\lambda(x', u, z, z')$;

(ii) the only witnesses z' for each of the four assignments $(x' = 1, u = 1, z = 0), (x' = 1, u = 0, z = 0), (x' = 1, u = 1, z = 1)$ and $(x' = 0, u = 0, z = 1)$ that satisfy the formula $(\exists z')\lambda(x', u, z, z')$ are $z' = 1, z' = 0, z' = 1$ and $z' = 1$, respectively.

Proof of Lemma 4.3

Let $\kappa'(x, y, z)$ be the formula $\psi_2(y, x, z)$ constructed in the last part of the proof of Lemma 4.2 (notice the inversion of x and y in ψ_2). From the properties of ψ_2, it immediately follows that κ' is satisfied by the truth assignments $(1, 1, 1)$, $(1, 0, 0)$ and $(0, 0, 1)$ but is not satisfied by the truth assignment $(1, 0, 1)$. To prove the properties (i)–(ii), we essentially do exhaustive case analysis for all the possible assignments to the variables x', z, u. We can immediately check that the formula $x' \equiv (u \vee \neg z)$ is satisfied by the assignments $(1, 1, 0), (1, 0, 0), (1, 1, 1)$ and $(0, 0, 1)$ (each bit in each assignment is assigned to x', u and z in this order), while it is not satisfied by the assignments $(0, 1, 0), (0, 0, 0), (0, 1, 1)$ and $(1, 0, 1)$. Now by plugging into the formula $(\exists z')\lambda(x', u, z, z')$ the latter four assignments, one after the other, we can check that they do not satisfy it. In the same way we can check that the former four assignments $(1, 1, 0), (1, 0, 0), (1, 1, 1)$ and $(0, 0, 1)$ do satisfy $(\exists z')\lambda(x', u, z, z')$. During the check that the above four assignments are indeed satisfying, we also determine all possibilities for the witness z', in order to verify that the uniqueness properties required from z' are indeed true (we will only need some of these uniqueness properties). ∎