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Cut Formulas for Kalmar Elementary Functions

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Abstract

We show that for every Kalmar elementary function we can find appropriate cut-formulas and using them we find proofs of length linear in the arguments of the function in predicate logic of the well definedness of the function. The usual proof of cut elimination shows the converse. This gives a new characterization of the Kalmar elementary functions as the functions which we can feasibly prove to be well defined in predicate logic using cut.

1 Main result

We consider classical predicate logic with equality. The language consists of a unary predicate $\mathcal{N}$, the constant 0, the unary successor function s and possibly a number of function symbols. As axioms we have the elementary successor axioms:

$$0 : \mathcal{N}$$
$$\forall x : \mathcal{N}.\ sx : \mathcal{N}$$

and a number of axioms defining the function symbols not involving the predicate $\mathcal{N}$ like:

$$\begin{aligned} +0y &= y \\ +sxy &= s + xy \\ \star 0y &= 0 \\ \star sxy &= +y \star xy \end{aligned}$$

These equational axioms are required to be true in the standard interpretation. A term txy is well defined if there are uniform proofs of the following statement from the axioms for all numerals m and n:

$$tmn : \mathcal{N}$$

The term is feasibly well defined if the proofs are short relative to m and n. We show that all Kalmar elementary terms [8] are feasibly well defined. The usual cut elimination theorem for classical predicate logic [9] shows the converse — any feasibly well defined term is Kalmar elementary. Furthermore we can from the definition of the term easily construct the short proofs and they will be linear in the arguments using cut formulas with complexity bound calculated from the compositions and the bounded sums and products involved in the definition of the term.

The arguments are carried out for classical predicate logic with equality. A similar result holds for classical predicate logic without equality.

2 Relevance

It is well known that the cut rule:

$$\varphi \vee \psi \ , \ \varphi \vee \neg\psi \Rightarrow \varphi$$

is crucial in logic programming. In the usual systems for logic programming we only allow propositional cut formulas ψ. Orevkov [6] and Zhang [10] have studied terms built up from exponentiation ($exy = 2^x + y$). We have as equational axioms:

$$\begin{aligned} e0y &= sy \\ esxy &= exexy \end{aligned}$$

Using appropriate cuts they can give a one page proof that:

$$eeeeeeee000000000 : \mathcal{N}$$

But if only propositional cuts are allowed, then the size of the proof exceeds the universe.

In proof theory there has been a lot of effort in finding methods to eliminate cuts. Very little was done in trying to find appropriate cuts making proofs feasible. We should learn how to introduce cuts, and not to eliminate them.

Cut formulas can be thought of as the auxiliary lemmas needed in proofs to make the proofs feasible. In logic programming we may also think of the cuts as giving the interaction part in interactive logic programming.

We show that there is a systematic procedure from the definition of a Kalmar elementary function to find the cuts making the proofs that it is well defined feasible. Using the ordinary cut elimination theorem we also get the converse.

3 Technical development

The crucial new concepts are inductive predicates and inductive terms defined as follows.

Definition. *A unary predicate $\mathcal{P}$ is inductive if it is similar to $\mathcal{N}$, $0 : \mathcal{P}$ and $\forall x : \mathcal{P}.\ sx : \mathcal{P}$ are provable. A term txy is inductive with x as induction variable if there are inductive predicates $\mathcal{P}_0$ and $\mathcal{P}_1$ making the following inductive:*

$$x : \mathcal{P}_0 \wedge \forall y : \mathcal{P}_1.\ txy : \mathcal{N}$$

Similar for terms with one or more arguments.

It is straightforward to prove that the following is inductive:

$$\begin{aligned} x : \mathcal{A} &= x : \mathcal{N} \wedge \forall y : \mathcal{N}.\ +xy : \mathcal{N} \\ x : \mathcal{M} &= x : \mathcal{N} \wedge \forall y : \mathcal{A}.\ \star xy : \mathcal{N} \\ x : \mathcal{E} &= x : \mathcal{N} \wedge \forall y : \mathcal{N}.\ exy : \mathcal{N} \end{aligned}$$

This shows that addition, multiplication and exponentiation are inductive. We must be a little careful to show that modified subtraction is inductive. To this we need the following equational axioms:

$$\begin{aligned} \dot{-}0y &= 0 \\ \dot{-}sxy = 0 &\vee \dot{-}sxy = s\dot{-}xy \end{aligned}$$

We use x and not y as induction variable. This is the only formula involving $\vee$ in our theory. Everything else involves only $\wedge, \rightarrow, \forall$. We then get that modified subtraction is inductive:

$$x : \mathcal{S} = x : \mathcal{N} \wedge \forall y : \mathcal{N}.\ \dot{-}xy : \mathcal{N}$$

To do actual calculations involving modified subtraction we add the usual recursion equations for predecessor p and modified subtraction:

$$\begin{aligned} p0 &= 0 \\ psx &= x \\ \dot{-}x0 &= x \\ \dot{-}xsy &= p\dot{-}xy \end{aligned}$$

Assume we have inductive predicates $\mathcal{F}$ and $\mathcal{G}$. The conjunction $\mathcal{F} \wedge \mathcal{G}$ is defined as usual. The composition $\mathcal{F}[\mathcal{G}]$ is defined by substituting the predicate $\mathcal{G}$ for $\mathcal{N}$ in $\mathcal{F}$. The inductive predicates are obviously closed under conjunction. Since the only axioms involving $\mathcal{N}$ are similar to the proved formulas of the inductive predicates, we also get that inductive predicates are closed under composition.

For composition of inductive terms assume we have the following inductive predicates:

$$x : \mathcal{F} = x : \mathcal{F}_0 \wedge \forall y : \mathcal{F}_1 .\, fxy : \mathcal{N}$$
$$u : \mathcal{G} = u : \mathcal{G}_0 \wedge \forall v : \mathcal{G}_1 .\, guv : \mathcal{N}$$

Then the following are also inductive:

$$x : \mathcal{F}_0 \wedge \forall u : \mathcal{G}[\mathcal{F}_1].\, \forall v : \mathcal{G}_1[\mathcal{F}_1].\, fxguv : \mathcal{N}$$
$$u : \mathcal{G}_0[\mathcal{F}] \wedge \forall v : \mathcal{G}_1[\mathcal{F}].\, \forall y : \mathcal{F}_1 .\, fguvy : \mathcal{N}$$

For diagonalization we start with the inductive predicate:

$$x : \mathcal{F}_0 \wedge \forall y : \mathcal{F}_1 .\, \forall z : \mathcal{F}_2 .\, fxyz : \mathcal{N}$$

Then the following are also inductive:

$$x : \mathcal{F}_0 \wedge \forall y : \mathcal{F}_1 \wedge \mathcal{F}_2 .\, fxyy : \mathcal{N}$$
$$x : (\mathcal{F}_0 \wedge \mathcal{F}_1) \wedge \forall z : \mathcal{F}_2 .\, fxxz : \mathcal{N}$$

For bounded sums and products we start with the inductive predicates $\mathcal{A}$ and $\mathcal{M}$ for addition and multiplication and the inductive predicate:

$$x : \mathcal{F} = x : \mathcal{F}_0 \wedge \forall y : \mathcal{F}_1 .\, fxy : \mathcal{N}$$

Then the following are also inductive:

$$z : \mathcal{F}[\mathcal{A}] \wedge \forall y : \mathcal{F}_1[\mathcal{A}].\, \Sigma_{x<z} fxy : \mathcal{N}$$
$$z : \mathcal{F}[\mathcal{M}] \wedge \forall y : \mathcal{F}_1[\mathcal{M}].\, \Pi_{x<z} fxy : \mathcal{N}$$
$$z : \mathcal{F}_1[\mathcal{A}] \wedge \forall x : \mathcal{F}[\mathcal{A}].\, \Sigma_{y<z} fxy : \mathcal{N}$$
$$z : \mathcal{F}_1[\mathcal{M}] \wedge \forall x : \mathcal{F}[\mathcal{M}].\, \Pi_{y<z} fxy : \mathcal{N}$$

Theorem. *All Kalmar elementary terms are inductive.*

This gives immediately the main result. Furthermore the inductive formulas are constructed from the basic inductive formulas $\mathcal{N}$ using conjunction, composition and universal quantification over inductive formulas and terms from the language.

As cut formulas in the proof of feasibly well definedness of a term we use the inductive formulas which we get in the construction of the term.

Lastly we come to the equality. This has often been a stumbling block for cut elimination. Here there are three ways to proceed

- in the usual formulation of sequential calculus with equality we can eliminate all non-atomic cuts
- Negri and von Plato [5] has given a formulation of sequential calculus with equality that admits full cut elimination

- avoid equality by writing the equality rules as appropriate implications

The first way is well known and does not merit further comment. For the second way we have the following sequential rules for equality [5]

$$\frac{a = a, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}$$

$$\frac{P(b/x), a = b, P(a/x), \Gamma \Rightarrow \Delta}{a = b, P(a/x), \Gamma \Rightarrow \Delta}$$

In this system we have full cut elimination. For the third way we observe that all the rewriting coming from the equational axioms are in one direction — from left to right. This means that we can replace the use of equality with appropriate implications. This is simplest done if we first replace the function symbols with relation symbols. E g for addition and multiplication we introduce two ternary relation symbols A and M and have as axioms:

$$\begin{gathered} A0yy \\ Axyz \rightarrow Asxysz \\ M0y0 \\ Mxyz \wedge Ayzu \rightarrow Msxyu \end{gathered}$$

We can translate our results into these frameworks.

4 Comparisons

The starting point has been the examples of Orevkov [6] and Zhang [10]. They showed that the terms built up from exponentiation could be given a short proof of well definedness using appropriate cut formulas. The usual cut elimination theorem gives a super exponential jump in length. We have designed our system such that there are no cut free proofs of well definedness of a closed term of length less than the numerical value of the term. If we use propositional cuts we may gain an exponential speed up in the length, but not more.

Holden [2, 3] carried the work further by giving a characterization of the Kalmar elementary terms as exactly those which can be given feasible proofs. But in her work the system used is not ordinary predicate logic, but an infinitary system with uniformity conditions.

Bellantoni and Cook [1] has given a system for polytime arithmetic using a restricted form of induction. They differentiated between induction arguments and substitution arguments — calling them normal and safe arguments. Our induction variables are similar to normal arguments — we are not free to substitute arbitrary values in them. Leivant [4] generalized this to a tiered hierarchy and went from the polytime functions to the Kalmar elementary functions. There is also recent work by Ostrin [7]. From their work a theorem like our main theorem was expected.

Our results give concrete feasible proofs using cut formulas. As an example consider the factorial. An inductive formula for the factorial f is:

$$\begin{array}{rc} & x:\mathcal{M} \wedge fx:\mathcal{N} \\ \Leftrightarrow & \forall y.\, (y:\mathcal{A} \rightarrow \star xy:\mathcal{N}) \wedge fx:\mathcal{N} \\ \Leftrightarrow & \forall y.\, (\forall z:\mathcal{N}.\ +yz:\mathcal{N} \rightarrow \star xy:\mathcal{N}) \wedge fx:\mathcal{N} \end{array}$$

And as cutformulas to get a short proof that the factorial is well defined we use subformulas of it. Similar calculations can be done for all Kalmar elementary functions. Our framework can also be used to define the Ackermann function a and the exponential tower function t:

$$\begin{array}{rcl} a0y & = & sy \\ asx0 & = & axs0 \\ asxsy & = & axasxy \\ t0 & = & 0 \\ tsx & = & etx0 \end{array}$$

But there are no short proofs in predicate logic that they are well defined.

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