THEORIES AND ORDINALS IN PROOF THEORY

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Theories and Ordinals in Proof Theory*

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1 Introduction

A central theme running through proof theory is the classification of theories by means of ordinals. This is manifest in the assignment of 'proof theoretic ordinals' to theories, gauging their 'consistency strength' and 'computational power'. To put it roughly, such ordinal analyses attach ordinals in a given representation system to formal theories.

The present paper gathers together results which explain the nature of the connection between ordinal representation systems and theories established in ordinal analyses by a more semantical approach in that it characterizes these ordinals in terms of familiar notions from model theory, set theory, and generalized recursion theory.

2 Measures in proof theory

2.1 Gentzen's result

Gentzen showed that transfinite induction up to the ordinal

$$\varepsilon_0 \ = \ \sup\{\omega, \omega^{\omega}, \omega^{\omega^{\omega}}, \ldots\} \ = \ \text{least } \alpha.\ \omega^{\alpha} = \alpha$$

suffices to prove the consistency of Peano Arithmetic, **PA**. To appreciate Gentzen's result it is pivotal to note that he applied transfinite induction up to ε_0 solely to primitive recursive predicates and besides that his proof used only finitistically justified means. Hence, a more precise rendering of Gentzen's result is

$$\mathbf{F} + \text{PR-TI}(\varepsilon_0) \vdash \mathsf{Con}(\mathbf{PA}), \tag{1}$$

where $\mathbf{F}$ signifies a theory that is acceptable in finitism (e.g. $\mathbf{F} = \mathbf{PRA}$ = *Primitive Recursive Arithmetic*) and PR-TI(ε_0) stands for transfinite induction up to ε_0 for

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primitive recursive predicates. Gentzen also showed that his result is best possible in that **PA** proves transfinite induction up to α for arithmetic predicates for any $\alpha < \varepsilon_0$. The compelling picture conjured up by the above is that the non-finitist part of **PA** is encapsulated in PR-TI(ε_0) and therefore "measured" by ε_0, thereby tempting one to adopt the following definition of *proof-theoretic ordinal* of a theory T:

$$|T|_{Con} \;=\; \text{least } \alpha.\ \mathbf{PRA} + \text{PR-TI}(\alpha) \vdash \mathsf{Con}(\mathbf{T}). \tag{2}$$

The foregoing definition of $|T|_{Con}$ is, however, inherently vague because the following issues have not been addressed:

- How are ordinals to be represented in **PRA**?
- (2) is definitive only with regard to a prior choice of *ordinal representation system.*
- Different ordinal representation systems may provide different answers to (2).

Notwithstanding that, for "natural" theories T and with regard to a "natural" ordinal representation system, the ordinal $|T|_{Con}$ encapsulates important information about the proof strength of T.

The next section will introduce a notion of proof-theoretic ordinal, $|T|_{\sup}$, which does not hinge on the choice of a particular ordinal representation system.

3 The general form of an ordinal analysis

In this section I attempt to say something general about all ordinal analyses that have been carried out thus far. One has to bear in mind that these concern "natural" theories. Also, to circumvent countless and rather boring counter examples, I will only address theories that have at least the strength of Primitive Recursive Arithmetic, **PRA**.

3.1 Theories

Ordinal analysis is concerned with theories serving as frameworks for formalising parts of mathematics. It is known that virtually all of ordinary mathematics can be formalized in Zermelo-Fraenkel set theory with the axiom of choice, **ZFC**. Hilbert and Bernays [11] showed that large chunks of mathematics can already be formalized in second order arithmetic. Owing to these observations, proof theory has been focusing on set theories and subsystems of second order arithmetic.

3.1.1 Subsystems of second order arithmetic

The language $\mathcal{L}_2$ of second-order arithmetic contains (free and bound) number variables $a, b, c, \ldots, x, y, z, \ldots$, (free and bound) set variables $A, B, C, \ldots, X, Y, Z, \ldots$, the constant 0, function symbols $Suc, +, \cdot$, and relation symbols $=, <, \in$. Suc stands for the successor function.

Terms are built up as usual. For $n \in \mathbf{N}$, let $\bar{n}$ be the canonical term denoting n. Formulae are built from the prime formulae $s = t$, $s < t$, and $s \in A$ using $\wedge, \vee, \neg, \forall x, \exists x, \forall X$ and $\exists X$ where s, t are terms.

Note that equality in $\mathcal{L}_2$ is only a relation on numbers. However, equality of sets will be considered a defined notion, namely

$$A = B \quad \text{iff} \quad \forall x[x{\in}A \leftrightarrow x{\in}B].$$

As usual, number quantifiers are called bounded if they occur in the context $\forall x(x < s \to \dots)$ or $\exists x(x < s \wedge \dots)$ for a term s which does not contain x. The $\mathbf{\Delta}^0_0$-formulae are those formulae in which all quantifiers are bounded number quantifiers, $\mathbf{\Sigma}^0_k$-formulae are formulae of the form $\exists x_1 \forall x_2 \dots Qx_k F$, where F is $\mathbf{\Delta}^0_0$, $\mathbf{\Pi}^0_k$-formulae are those of the form $\forall x_1 \exists x_2 \dots Qx_k F$. The union of all $\mathbf{\Pi}^0_k$- and $\mathbf{\Sigma}^0_k$-formulae for all $k \in \mathbf{N}$ is the class of *arithmetical* or $\mathbf{\Pi}^0_\infty$*-formulae.* The $\mathbf{\Sigma}^1_k$-formulae ($\mathbf{\Pi}^1_k$-formulae) are the formulae $\exists X_1 \forall X_2 \dots QX_k F$ (resp. $\forall X_1 \exists X_2 \dots Qx_k F$) for arithmetical F.

The basic axioms in all theories of second-order arithmetic are the defining axioms of $0, 1, +, \cdot, <$ and the *induction axiom*

$$\forall X(0 \in X \wedge \forall x(x \in X \to x+1 \in X) \to \forall x(x \in X)),$$

respectively the *schema of induction*

$$\textbf{IND} \qquad F(0) \wedge \forall x(F(x) \to F(x+1)) \to \forall x F(x),$$

where F is an arbitrary $\mathcal{L}_2$-formula.

We consider the axiom schema of $\mathcal{C}$*-comprehension* for formula classes $\mathcal{C}$ which is given by

$$\mathcal{C} - \textbf{CA} \qquad \exists X \forall u(u \in X \leftrightarrow F(u))$$

for all formulae $F \in \mathcal{C}$ in which X does not occur.

For each axiom schema $\textbf{Ax}$ we denote by $(\textbf{Ax})$ the theory consisting of the basic arithmetical axioms, the schema $\mathbf{\Pi}^0_\infty - \textbf{CA}$, the schema of induction and the schema $\textbf{Ax}$. If we replace the schema of induction by the induction axiom, we denote the resulting theory by $(\textbf{Ax})\restriction$.

An example for these notations is the theory $(\mathbf{\Pi}^1_1 - \textbf{CA})$ which contains the induction schema, whereas $(\mathbf{\Pi}^1_1 - \textbf{CA})\restriction$ only contains the induction axiom in addition to the comprehension schema for $\mathbf{\Pi}^1_1$-formulae.

In the framework of these theories one can introduce defined symbols for all primitive recursive functions. Especially, let $\langle , \rangle : \mathbb{N} \times \mathbb{N} \longrightarrow \mathbb{N}$ be a primitive recursive and bijective pairing function.

The x^{th} section of U is defined by $U_x := \{y : \langle x, y\rangle{\in}U\}$. Observe that a set U is uniquely determined by its sections on account of $\langle , \rangle$'s bijectivity.

Any set R gives rise to a binary relation $\prec_R$ defined by $y \prec_R x := \langle y, x\rangle{\in}R$.

Using the latter coding, we can formulate the axiom of choice for formulae F in $\mathcal{C}$ by

$$\mathcal{C} - \textbf{AC} \qquad \forall x \exists Y F(x, Y) \to \exists Y \forall x F(x, Y_x).$$

A special form of comprehension is $\mathbf{\Delta}^1_n$-comprehension, that is

$$\mathbf{\Delta}^1_n - \textbf{CA} \qquad \forall u\big[\phi(u) \leftrightarrow \vartheta(u)\big] \to \exists X \forall u(u{\in}X \leftrightarrow \phi(u))$$

for all $\mathbf{\Pi}^1_n$-formula ϕ and $\mathbf{\Sigma}^1_n$-formula ϑ.

Bar induction is the schema

$$\textbf{BI} \qquad \forall X\big[\mathrm{WF}(\prec_X) \;\wedge\; \forall u\big(\forall v \prec_X u \phi(v) \to \phi(u)\big) \;\to\; \forall u \phi(u)\big]$$

for all formulae ϕ, where $\mathrm{WF}(\prec_X)$ expresses that $\prec_X$ is well-founded (see Definition 3.2).

3.1.2 Subsystems of set theory

The axiom systems for set theory considered in this paper are formulated in the usual language of set theory (called $\mathcal{L}_\in$ hereafter) containing $\in$ as the only non-logical symbol besides $=$. Formulae are built from prime formulae $a \in b$ and $a = b$ by use of propositional connectives and quantifiers $\forall x, \exists x$. Bounded quantifiers $\forall x \in a$, $\exists x \in a$ are defined as usual. Δ_0-formulae are the formulae wherein all quantifiers are bounded; Σ_1-formulae are those of the form $\exists x \varphi(x)$ where $\varphi(a)$ is a Δ_0-formula. For $n > 0$, Π_n-formulae (Σ_n-formulae) are the formulae with a prefix of n alternating unbounded quantifiers starting with a universal (existential) one followed by a Δ_0-formula. The class of Σ-formulae is the smallest class of formulae containing the Δ_0-formulae which is closed under $\wedge$, $\vee$, bounded quantification and unbounded existential quantification.

One of the set theories which is amenable to ordinal analysis is Kripke-Platek set theory, **KP**. Its standard models are called *admissible sets*. One of the reasons that this is a truly remarkable theory is that a great deal of set theory requires only the axioms of **KP**. An even more important reason is that admissible sets have been a major source of interaction between model theory, recursion theory and set theory (cf. [2]). **KP** arises from **ZF** by completely omitting the power set axiom and restricting separation and collection to absolute predicates (cf. [2]), i.e. Δ_0 formulas. These alterations are suggested by the informal notion of 'predicative'.

Definition 3.1 The axioms of **KP** are:

Extensionality:	$\forall x(x{\in}a \leftrightarrow x{\in}b) \rightarrow a = b$.
Foundation:	$\forall x[(\forall y{\in}x)G(y) \rightarrow G(x)] \rightarrow \forall x G(x)$ for all formulae G.
Pair:	$\exists x\ (x = \{a, b\})$.
Union:	$\exists x\ (x = \bigcup a)$.
Infinity:	$\exists x\ \big[x \neq \emptyset\ \wedge\ (\forall y{\in}x)(\exists z{\in}x)(y{\in}z)\big]$.[1]
Δ_0 *Separation:*	$\exists x\ \big(x = \{y{\in}a : F(y)\}\big)$[2] for all Δ_0–formulas F in which x does not occur free.
Δ_0 *Collection:*	$(\forall x{\in}a)\exists y G(x,y) \rightarrow \exists z(\forall x{\in}a)(\exists y{\in}z)G(x,y)$ for all Δ_0–formulas G.

L_α, the αth level of Gödel's constructible hierarchy L, is defined by $L_0 = \emptyset$, $L_{\beta+1} = \big\{X : X \subseteq L_\beta;\ X \text{ definable over } \langle L_\beta, \in\rangle\big\}$ and $L_\lambda = \bigcup\{L_\beta : \beta < \lambda\}$ for limits λ. So any element of L of level α is definable from elements of L with levels $< \alpha$ and L_α.

A transitive set A such that $(A, \in)$ is a model of **KP** is called an *admissible set*. An ordinal α is *admissible* if the structure $(L_\alpha, \in)$ is a model of **KP**.

Some systems of set theories will be used later for illustrative purposes. **KPi** is an extension of **KP** via the axiom

$$(Lim) \qquad \forall x \exists y[x{\in}y \wedge y \textit{ is an admissible set}].$$

[1]This contrasts with [2] where Infinity is not included in **KP**.

[2]$x = \{y{\in}a : F(y)\}$ stands for the Δ_0–formula $(\forall y{\in}x)[y{\in}a \wedge F(y)] \wedge (\forall y{\in}a)[F(y) \rightarrow y{\in}x]$.

KPl denotes the system **KPi** without Δ_0 Collection. $\mathbf{KPl}^r$ and $\mathbf{KPi}^r$ arise from **KPl** and **KPi**, respectively, by restricting the scheme of Foundation to Δ_0 formulae G. $\mathbf{KPi}^w$ is obtained from $\mathbf{KPi}^r$ by adding the schema

$$\mathbf{IND}_\omega \qquad \forall x \in \omega(\forall y \in x F(y) \to F(x)) \to \forall x \in \omega F(x)$$

of induction on ω for all formulae ϕ.

The foregoing set theories are closely related to well studied subsystems of second order arithmetic. $\mathbf{KPl}^r$, **KPl**, $\mathbf{KPi}^w$, and **KPi** prove the same sentences of second order arithmetic as $(\mathbf{\Pi}^1_1-\mathbf{CA})\restriction$, $(\mathbf{\Pi}^1_1-\mathbf{CA})+\mathbf{BI}$, $(\mathbf{\Delta}^1_2-\mathbf{CA})$, and $(\mathbf{\Delta}^1_n-\mathbf{CA})+\mathbf{BI}$, respectively.

3.2 $|T|_{\sup}$

Before delineating the general form of an ordinal analysis, we need several definitions.

Definition 3.2 Let T be a framework for formalizing a certain part of mathematics. T should be a true theory which contains a modicum of arithmetic.

Let A be a subset of $\mathbb{N}$ ordered by $\prec$ such that A and $\prec$ are both definable in the language of T. If the language of T allows for quantification over subsets of $\mathbb{N}$, like that of second order arithmetic or set theory, *well-foundedness* of $\langle A, \prec\rangle$ will be formally expressed by

$$\mathrm{WF}(A,\prec) \;:=\; \forall X \subseteq \mathbb{N}[\forall u{\in}A(\forall v \prec u\, v{\in}X \to u{\in}X) \;\to\; \forall u{\in}A\, u{\in}X.] \qquad (3)$$

If, however, the language of T does not provide for quantification over arbitrary subsets of $\mathbb{N}$, like e.g. that of Peano arithmetic, we shall assume that it contains a new unary predicate **U**. **U** acts like a free set variable, in that no special properties of it will ever be assumed. We will then resort to the following formalization of well-foundedness:

$$\mathrm{WF}(A,\prec) \;:=\; \forall u{\in}A(\forall v \prec u\, \mathbf{U}(v) \to \mathbf{U}(u)) \;\to\; \forall u{\in}A\, \mathbf{U}(u), \qquad (4)$$

where $\forall v \prec u \ldots$ is short for $\forall v(v \prec u \to \ldots)$. We shall use $\mathrm{WF}(\prec)$ as an abbreviation for $\mathrm{WF}(\mathbb{N},\prec)$. We also set

$$\mathrm{WO}(A,\prec) \;:=\; \mathrm{LO}(A,\prec) \wedge \mathrm{WF}(A,\prec). \qquad (5)$$

If $\langle A,\prec\rangle$ is well-founded, we use $|\prec|$ to signify its set-theoretic order-type. For $a{\in}A$, the ordering $\prec\restriction a$ denotes the restriction of $\prec$ to $\{x{\in}A : x \prec a\}$.

The ordering $\langle A,\prec\rangle$ is said to be *provably well-founded in T* if

$$T \vdash \mathrm{WO}(A,\prec). \qquad (6)$$

The supremum of the provable well-orderings of T, $|T|_{\sup}$, is defined as follows:

$$|T|_{\sup} \;:=\; \sup\{\alpha : \alpha \text{ provably recursive in } T\} \qquad (7)$$

where an ordinal α is said to be provably recursive in T if there is a recursive well–ordering $\langle A,\prec\rangle$ with order–type α such that

$$T \vdash \mathrm{WO}(A,\prec)$$

with A and $\prec$ being provably recursive in T. Note that, by definition, $|T|_{\sup} \le \omega_1^{CK}$, where ω_1^{CK} is the supremum of the order-types of all recursive well-orderings on $\mathbb{N}$. Another characterization of ω_1^{CK} is that it is the least admissible ordinal $> \omega$.

Agreement. From now on the *proof-theoretic ordinal* of a theory T is taken to be $|T|_{\sup}$.

3.3 The robustness of $|T|_{\sup}$

This subsection gathers together several results which show that there is a lot of leeway in defining $|T|_{\sup}$. Instead of recursive well-orderings we could have restricted ourselves to primitive recursive or even elementary recursive well-orderings. On the other hand it is also possible to go into the other direction by allowing for well-orderings of greater complexity.

The statements below involve certain well known subsystems of **PA** and second order arithmetic. $\mathbf{I\Sigma_1}$ denotes the fragment of **PA** obtained by restricting induction to Σ_1 formulas. $\mathbf{WKL_0}$ is a fragment of second order arithmetic whose main set existence axiom is a version of König's lemma restricted to binary trees. $\mathbf{WKL_0}$ is proof-theoretically of the same strength as $\mathbf{I\Sigma_1}$, and thus weaker than **PA**. For an excact definition and the role of of these theories in the program of *Reverse Mathematics* see [30].

Proposition 3.3 *(i) Suppose that for every elementary well-ordering $\langle A, \prec \rangle$, whenever $T \vdash \mathrm{WO}(A, \prec)$, then*

$$T \vdash \forall u\,[\,A(u) \to (\forall v \prec u P(v)) \to P(u)] \to \forall u\,[A(u) \to P(u)]$$

holds for all provably recursive predicates P of T. Then

$$\begin{aligned} |T|_{\sup} &= \sup\{\alpha : \ \alpha \text{ is provably elementary in } T\} &(8)\\ &= \sup\{\alpha : \ \alpha \text{ is provably } \Sigma^0_1 \text{ in } T\}. \end{aligned}$$

Moreover, if $T \vdash \mathrm{WO}(A, \prec)$ and *$A, \prec$ are provably recursive in T, then one can find an elementary well-ordering $\langle B, \lessdot \rangle$ and a recursive function f such that $T \vdash \mathrm{WO}(B, \lessdot)$, f is provably recursive in T, and T proves that f supplies an order isomorphism between $\langle B, \lessdot \rangle$ and $\langle A, \prec \rangle$.*

Examples for (i) are the theories $\mathbf{I\Sigma_1}$, $\mathbf{WKL_0}$ *and* **PA**.

(ii) If T comprises $(\mathbf{\Pi}^0_\infty - \mathbf{CA})\!\restriction$, then

$$|T|_{\sup} = \sup\{\alpha : \ \alpha \text{ is provably arithmetic in } T\}. \tag{9}$$

(iii) If T comprises $(\mathbf{\Sigma}^1_1 - \mathbf{AC})\!\restriction$, then

$$|T|_{\sup} = \sup\{\alpha : \ \alpha \text{ is provably analytic in } T\}, \tag{10}$$

where a relation on $\mathbb{N}$ is called analytic *if it is lightface Σ^1_1.*

Proof: [24], Proposition 2.19. □

A theory is said to be Π^1_1-faithful if all of its theorems of complexity Π^1_1 are true.

Theorem 3.4 *Let T be a Σ^1_1 axiomatizable theory.*

(i) If T is Π^1_1-faithful, then $|T|_{\sup} < \omega_1^{CK}$.

(ii) If $(\mathbf{\Pi}^0_\infty - \mathbf{CA})\!\restriction \subseteq T$ and $|T|_{\sup} < \omega_1^{CK}$, then T is Π^1_1-faithful.

(iii) There are consistent primitive recursive theories T such that $|T|_{\sup} = \omega_1^{CK}$.

Proof: See [24], Theorem 2.4. $\square$

Another feature of of $|T|_{\sup}$ is that this ordinal does not change when one augments T by true Σ^1_1 statements.

Proposition 3.5 *Let T be a primitive recursive, Π^1_1-faithful theory of second order arithmetic such that* $\mathbf{PA} \subseteq \mathbf{T}$. *Let $\lhd$ be a primitive recursive well-ordering such that $|T|_{\sup} = |\lhd|$ and*

$$\mathbf{PA} + \mathrm{TI}(\lhd) \vdash \mathsf{Proof}_{\mathbf{T}}(\ulcorner \mathbf{F} \urcorner) \rightarrow \mathbf{F} \tag{11}$$

holds for all arithmetic formulae F which may contain free second order set variables but no free number variables. Then, for any true Σ^1_1 statement B,

$$|T|_{\sup} = |T + B|_{\sup}.$$

Proof: See [24], Proposition 2.6. $\square$

Remark 3.6 In all the examples I know, if T is a subsystem of classical second order arithmetic for which an ordinal analysis has been carried out via an ordinal representation system $(A, \lhd)$, (11) is satisfied.

4 Model-theoretic characterizations

This first part of this section shows that $|T|_{\sup}$ can be couched in terms of partial models in the constructible hierarchy. The second part presents Carlson's approach of obtaining ordinal representations from finite structures.

4.1 Partial models

Recall that L_α, the αth level of Gödel's constructible hierarchy L, is defined by $L_0 = \emptyset$, $L_{\beta+1} = \{X : X \subseteq L_\beta;\ X \text{ definable over } \langle L_\beta, \in \rangle\}$ and $L_\lambda = \bigcup\{L_\beta : \beta < \lambda\}$ for limits λ. So any element of L of level α is definable from elements of L with levels $< \alpha$ and L_α.

Definition 4.1 For a collection of sentences (in the language of set theory), $\mathcal{F}$, we say that L_α is an *$\mathcal{F}$-model of T* if for all $B \in \mathcal{F}$, whenever $T \vdash B$, then $L_\alpha \models B$. Let

$$|T|_{\mathcal{F}} \ := \ \min\{\alpha : \ L_\alpha \text{ is an } \mathcal{F}\text{-model of } T\}.$$

Definition 4.2 Let $\mathcal{F}$ be a collection of sentences. A set theory T is said to be *$\mathcal{F}$-sound* if for every $\mathcal{F}$ theorem ϕ of T, $L \models \phi$ holds.

For a collection of formulae $\mathcal{F}$, let $\mathcal{F}(L_\alpha)$ consist of all formulae A^{L_α} with $A \in \mathcal{F}$.

The system **PRST** (for *Primitive Recursive Set Theory*) is formulated in the language of set theory augmented by symbols for all primitive recursive set functions in the sense of Jensen and Karp [15]. The *axioms of* **PRST** are Extensionality, Pair, Union, Infinity, Δ_0-Separation, the Foundation Axiom (i.e. $x \neq \emptyset \rightarrow (\exists y \in x)(\forall z \in y)\, z \notin x$) and the defining equations for the primitive recursive set functions.

In the following we shall assume that all set theories contain **PRST** either directly or via interpretation.

The next theorem gives a characterization of the proof-theoretic ordinal of T in terms of $|T|_{\mathcal{F}}$ for two classes of formulae. It requires, however, that T proves the existence of ω_1^{CK}. Recall that ω_1^{CK} stands for the least admissible ordinal $> \omega$. There is a canonical Π_3-sentence θ of set theory such that for every $\alpha > 0$, $L_\alpha \models \theta$ iff L_α is an admissible set (cf. [25]). We will say that T proves the existence of ω_1^{CK} if $T \vdash \exists\alpha > \omega\, \theta^{L_\alpha}$.

Theorem 4.3 *If T is Π_2-sound and and T proves the existence of ω_1^{CK}, then*

$$|T|_{\sup} = |T|_{\Sigma_1(L(\omega_1^{CK}))} = |T|_{\Pi_2(L(\omega_1^{CK}))}.$$

Proof: The equality $|T|_{\sup} = |T|_{\Sigma_1(L(\omega_1^{CK}))}$ follows from [19], Theorem 7.14.

$|T|_{\Sigma_1(L(\omega_1^{CK}))} = |T|_{\Pi_2(L(\omega_1^{CK}))}$ is an immediate consequence of the proof of [20], Theorem 2.1. □

An ordinal analysis of T also allows one to determine the ordinals $|T|_{\Sigma_1}$ and $|T|_{\Pi_2}$. This will be addressed in more detail in the last section. In point of fact, these ordinals are the same if T satisfies some mild requirements.

Proposition 4.4 *Suppose T is Π_2 sound and comprises Δ_0-collection. Furthermore, suppose that $T \vdash B$ implies $T \vdash \exists\alpha\exists x(x = L_\alpha \wedge B^x)$ for all Σ_1-sentences B. If T has a Σ_1-model then T has a Π_2-model and*

$$|T|_{\Sigma_1} = |T|_{\Pi_2}. \qquad (12)$$

Proof: [20]. Theorem 2.1. □

There are theories where $|T|_{\Sigma_1}$ and $|T|_{\sup}$ coincide. A prominent example is **KP**. The ordinal $\psi_\Omega(\varepsilon_{\Omega+1})$ is known as the *Bachmann-Howard ordinal.*

Theorem 4.5 $|\mathbf{KP}|_{\sup} = |\mathbf{KP}|_{\Sigma_1} = |\mathbf{KP}|_{\Pi_2} = \psi_\Omega(\varepsilon_{\Omega+1})$.

Proof: See [13] and [20]. □

4.2 Patterns of resemblance

An intriguing new way of defining ordinal representation systems has been pursued by Tim Carlson (cf. [6], [7]). In this approach the class of ordinals gets furnished with a relation of $\exists_1$ elementary substructurehood and the ordinal representations correspond to finite substructures of this class structure.

Definition 4.6 Suppose $\mathfrak{A} = (ORD, \ldots, \leq)$ is a (class) structure whose universe is the class of ordinals ORD, with the ordering $\leq$ of ordinals. A finite substructure $\mathfrak{F}$ of $\mathfrak{A}$ is said to be *isominimal* if there is no finite substructure $\mathfrak{F}'$ of $\mathfrak{A}$ such that

- $\mathfrak{F}' \equiv \mathfrak{F}$
- $\mathfrak{F}' \neq \mathfrak{F}$

- $\mathfrak{F}' \leq_{pw} \mathfrak{F}$,

where $\leq_{pw}$ denotes the *pointwise partial ordering* of finite sets of ordinals, i.e. $\mathfrak{F}' \leq_{pw} \mathfrak{F}$ iff both structures have the same number of elements and if $\alpha_0, \ldots, \alpha_{n-1}$ enumerates the elements of $\mathfrak{F}'$ in increasing order and $\beta_0, \ldots, \beta_{n-1}$ enumerates the elements of $\mathfrak{F}$ in increasing order then $\alpha_i \leq \beta_i$ for $i < n$.

Since the definition of Σ_1 formula in the usual set-theoretic sense allows arbitrary bounded quantifiers inside the initial existential quantifiers, we specify that a $\exists_1$ formula is a quantifier-free formula prefixed by a string of existential quantifications.

The *core* of $\mathfrak{A}$ is the union of the isominimal substructures of $\mathfrak{A}$.

[6] introduces a structure $\mathcal{R}_0$ whose core turns out to be the ubiquitous ordinal ε_0.

Definition 4.7 $\preceq_1^0$ is the partial ordering on the class of ordinals defined by induction so that

$$\alpha \preceq_1^0 \beta \quad \text{iff} \quad (\alpha, 0, \leq, \preceq_1^0) \text{ is a } \exists_1\text{-elementary substructure of } (\beta, 0, \leq, \preceq_1^0).$$

To be more precise, by induction on β we define the set of α such that $\alpha \preceq_1^0 \beta$ (note that we have taken some liberty in writing $(\alpha, \leq, \preceq_1^0)$ where we should have restricted the relations to α).

Theorem 4.8 (Carlson [6]) *The core of $\mathcal{R}_0$ is the ordinal ε_0.*

Augmenting the ordinals by the function of addition, [7] introduces a richer structure $\mathcal{R}_1$ whose core turns out to be the proof-theoretic ordinal of $(\mathbf{\Pi}_1^1 - \mathbf{CA})\restriction$.

Definition 4.9 $\preceq_1^1$ is the partial ordering on the class of ordinals defined by induction so that

$$\alpha \preceq_1^1 \beta \quad \text{iff} \quad (\alpha, 0, +, \leq, \preceq_1^1) \text{ is a } \exists_1\text{-elementary substructure of } (\beta, 0, +, \leq, \preceq_1^0).$$

It should be pointed out that contrary to standard practice, one allows structures to interpret $+$ as a partial operation on the universe, e.g. If $\beta, \gamma < \alpha$ but $\beta + \gamma \geq \alpha$ then $+$ is not defined for the arguments β, γ in the structure $(\alpha, 0, +, \leq, \preceq_1^1)$.

Theorem 4.10 (Carlson [7]) *The core of $\mathcal{R}_1$ is $\psi_{\Omega_1}\Omega_\omega$ (in the notation of [3]), the proof-theoretic ordinal of $(\mathbf{\Pi}_1^1 - \mathbf{CA})\restriction$.*

To give an idea of how the core of $\mathcal{R}_1$ gives rise to a recursive ordinal representation system we need some notions. A substructure $\mathfrak{B}$ of $\mathcal{R}_1$ is *closed* if $0 \in \mathfrak{B}$ and whenever $\omega^{\alpha_1} + \cdots \omega^{\alpha_m}$ is in $\mathfrak{B}$ with $\alpha_1 \geq \ldots \geq \alpha_m$ then $\omega^{\alpha_1}, \ldots, \omega^{\alpha_m} \in \mathfrak{B}$ and $\omega^{\alpha_1} + \cdots + \omega^{\alpha_i} \in \mathfrak{B}$ for $i = 1, \ldots, m$. Notice that every finite set of ordinals is contained in a finite set of ordinals which is closed.

It can be shown that for a fixed finite closed substructure $\mathfrak{F}$ of $\mathcal{R}_1$, there is a unique isominimal substructure $\mathfrak{F}^*$ of $\mathcal{R}_1$ which is isomorphic to $\mathfrak{F}$. Moreover, $\mathfrak{F}^*$ is closed. This provides a system of ordinal representations for the ordinals which occur in the core of $\mathcal{R}_1$: if α appears as the nth element of some closed isominimal substructure $\mathfrak{F}^*$ of $\mathcal{R}_1$ we can use the pair (τ, n) as a notation for α where τ is the isomorphism type of $\mathfrak{F}^*$. These notations allow one to show that the core of $\mathcal{R}_1$ is isomorphic to a recursive structure.

Remark 4.11 At first glance, the difference between the structures $\mathcal{R}_0$ and $\mathcal{R}_1$ seems only tiny as the operation of addition on ordinals appears to be innocent enough. A similar effect, however, has also been unearthed in a paper by Schütte and Simpson [29] wherein they show that omitting the operation $+$ from the ordinal representation system for the proof-theoretic ordinal of $(\mathbf{\Pi}_1^1 - \mathbf{CA})\restriction$ has a dramatic effect in that the order-type drops to ε_0.

Remark 4.12 Carlson has also considered richer structures than $\mathcal{R}_1$ whose cores are conjectured to provide ordinal representations for very strong subsystems of second order arithmetic.

5 Characterizations via E-recursion

A particular interesting measure that can be assigned to a set theory T is $|T|^E$, where the superscript E signifies E-*recursion*, also termed *set recursion*. E-recursion theory extends the notion of computation from the natural numbers to arbitrary sets. For details see [26].

Definition 5.1 The intent is to assign meaning to $\{e\}(x)$ for every set x via an appropriate notion of computation. The definition of $\{e\}(x)$ is in terms of schemes introduced by Norman [18], and subsequently and independently by Moschovakis [17]. The first three schemes are projection, difference and pairing. The fifth is composition, and the sixth is enumeration. Bounding with union, the fourth scheme, is the sole source of infinitely long computations. To be precise, E-recursion is defined by the following schemes:

1. $e = \langle 1, n, i\rangle$,

   $\{e\}(x_1, \ldots, x_n) = x_i$.

2. $e = \langle 2, n, i, j\rangle$,

   $\{e\}(x_1, \ldots, x_n) = x_i \backslash x_j$.

3. $e = \langle 3, n, i, j\rangle$,

   $\{e\}(x_1, \ldots, x_n) = \{x_i, x_j\}$.

4. $e = \langle 4, n, c\rangle$,

   $\{e\}(x_1, \ldots, x_n) = \bigcup\{\{c\}(y, x_2, \ldots, x_n) : y \in x_1\}$.

   The left side is not defined unless $\{c\}(y, x_2, \ldots, x_n)$ is defined for all $y \in x_1$.

5. $e = \langle 5, n, m, e', e_1, \ldots, e_n\rangle$,

   $\{e\}(x_1, \ldots, x_n) \simeq \{e'\}(\{e_1\}(x_1, \ldots, x_n), \ldots, \{e_m\}(x_1, \ldots, x_n))$.

6. $e = \langle 6, n, m\rangle$,

   $\{e\}(e_1, x_1, \ldots, x_n, y_1, \ldots, y_m) \simeq \{e_1\}(x_1, \ldots, x_n)$.

$\simeq$ is Kleene's symbol for strong equality. If g and f are partial functions, then $f(x) \simeq g(x)$ iff neither $f(x)$ nor $g(x)$ is defined, or $f(x)$ and $g(x)$ are defined and equal.

To some, enumeration is a theorem, not a scheme. Casting it as a scheme makes it possible to omit the least number operator and primitive recursion, two schemes well abandoned when there is no underlying effective wellordering of the sets.

Definition 5.2 A partial function from V, the class of all sets, into V is *partial E-recursive* if it belongs to the least class of partial functions closed under the Normann schemes. The graph of such a function is Σ_1 (in the language of set theory), the converse, however, does not hold. An example is $O(x)$, Gödel's order of constructibility function

$$O(x) \simeq \mu\gamma[x \in L_{\gamma+1} \setminus L_\gamma].$$

$O(x)$ is Σ_1 but not partial E-recursive. If $x \in L$ then $O(x)$ is found by an unbounded search devoid of effective content.

A theorem of van de Wiele [31] explains the gap between Σ_1 definability and partial E-recursiveness.

Definition 5.3 Let $f : V \to V$ be a total function. F is *uniformly Σ_1 definable on every admissible set* if there is a Σ_1 formula $\phi(x,y)$ (which contains at most the free variables exhibited) such that for every admissible set A:

- $(\forall x \in A)\, f(x) \in A$;
- $f \restriction A = \{\langle a,b\rangle : \langle A, \in\rangle \models \phi(a,b)\}$.

Theorem 5.4 (van de Wiele, [31]) *For every total function $f : V \to V$ the following are equivalent:*

(i) f is E-recursive.

(ii) f is uniformly Σ_1 definable on every admissible set.

Definition 5.5 The next notions are due to A. Schlüter [27].

$$|T|^E_{\Sigma_1} \;:=\; \min\{\alpha : \text{ for all } e \in \omega,\, T \vdash \{e\}(\omega)\downarrow \text{ implies } \{e\}(\omega) \in L_\alpha\}.$$

$$|T|_{\Pi^E_2} := \min\{\alpha > \omega : \text{ for all } e \in \omega,\, T \vdash \forall x\, \{e\}(x)\downarrow \text{ implies } \forall x \in L_\alpha \{e\}(x) \in L_\alpha\}.$$

For the remainder of this subsection it is assumed that all set theories comprise **PRST**.

Theorem 5.6 (Schlüter [27]) *If T is a Π_2 sound theory, then*

$$|T|^E_{\Sigma_1} \;=\; |T|^E_{\Pi_2}. \tag{13}$$

Proof: (13) is stated and proved in [27], 6.14. □

Theorem 5.7 (Schlüter [27]) *If T is Π_2-sound, then*

$$|T|_{\sup} = |T|^E_{\Sigma_1} = |T|^E_{\Pi_2}.$$

Proof: A detailed proof of $|T|^E_{\Sigma_1} = |T|_{\sup}$ can be found in [27], Satz 6.15. □

In point of fact, the proof of Theorem 5.6 also yields the following result.

Theorem 5.8 *If T is a Π_2 sound theory, then*

$$|T|_{\sup} = \{\alpha : \exists e{\in}\omega[\alpha = \{e\}(\omega) \ \wedge \ T \vdash \{e\}(\omega)\downarrow]\}.$$

6 The Σ_1 spectrum of a theory

The extraction of classifications of provable functions from ordinal analyses is not confined to recursive functions on natural numbers. In the case of fragments of second order arithmetic, one may also classify the provable hyperarithmetical as well as the provable Δ^1_2 functions on $\mathbb{N}$. In the case of set theories one may classify several kinds of provable set functions and ordinals.

In the following we will be concerned with norms that can be assigned to set theories. In general, they can also be extracted from an ordinal analysis of a set theory T. Among other results, they lead to a classification of the provable set functions of T.

Definition 6.1 Another notion that is closely related to the the norm $|T|_{\Sigma_1}$ is the notion of *good* Σ_1*-definition* from admissible set theory (see [2], II.5.13). Given a set theory T, we say that an ordinal α has a *good* Σ_1*-definition in* T if there is a Σ_1-formula $\phi(u)$ such that

$$L \models \phi[\alpha] \text{ and } T \vdash \exists!\xi\phi(\xi).$$

Let

$$\mathsf{spec}_{\Sigma_1} := \{\alpha : \ \alpha \text{ has a good } \Sigma_1 \text{ definition in } T\}.$$

One obviously has $\sup(\mathsf{spec}_{\Sigma_1}) = |T|_{\Sigma_1}$. In many cases the set spec_{Σ_1} bears interesting connections to the ordinals of the representation system that has been used to analyze T. Ordinal representation systems that have been developed via a detour through large cardinals allow for an alternative interpretation wherein the large cardinals are replaced by their recursively large counterparts. The latter interpretation gives rise to a canonical interpretation of the ordinal terms of the representation system in spec_{Σ_1}. In general, however, the ordinals of spec_{Σ_1} stemming from the ordinal representation form a proper subset of spec_{Σ_1} with many 'holes' as will be shown in the last section. It would be very desirable to find a 'natural' property which could distinguish the ordinals of the representation system within spec_{Σ_1} so as to illuminate their naturalness. I consider this to be one of the most important problems in the area of strong ordinal representation systems. A more thorough discussion will follow in section 4.

We will show that under very weak assumptions on $\mathbf{T}$ that the spectrum of $\mathbf{T}$ is an initial segment of the ordinals. Let $\mathbf{T}$ be a theory such that $\mathbf{T} \vdash \phi$ implies $L \models \phi$ for all Σ_1- and Π_1-sentences ϕ. We shall require that $\mathbf{T}$ contains a modicum of primitive recursive set theory, where by primitive recursive set theory we mean that all primitive recursive set function are provable in $\mathbf{T}$. In particular, we assume that the function $\eta \mapsto L_\eta$ is provable in $\mathbf{T}$. Moreover, we shall assume that $\mathbf{T} \vdash \forall\alpha\, \exists\lambda \geq \alpha[\lambda \text{ is a limit}]$. Then the following holds.

Theorem 6.2 (Möllerfeld, Rathjen [16])
$\mathsf{spec}_{\Sigma_1}(\mathbf{T})$ *is an ordinal, that is an initial segment of the ordinals.*

Proof: The proof makes use of the notion of stable ordinal. For an introduction to stable ordinals the reader is referred to Barwise's textbook, [2]. An ordinal η is *β-stable* if $\eta \leq \beta$ and L_η is a Σ_1-elementary substructure of L_β.

Set $\sigma_{\mathbf{T}} := \sup(\mathsf{spec}_{\Sigma_1}(\mathbf{T}))$. For each $\eta \in \mathsf{spec}_{\Sigma_1}(\mathbf{T})$ we pick a Σ_1-formula ϕ_η such that

$$\mathbf{T} \vdash \exists! \xi\, \phi_\eta(\xi) \quad \text{and} \quad L \models \phi_\eta[\eta]. \tag{14}$$

We shall proof, by induction on $\alpha < \sigma_{\mathbf{T}}$, that $\alpha \in \mathsf{spec}_{\Sigma_1}(\mathbf{T})$.
For this assume $\alpha \subseteq \mathsf{spec}_{\Sigma_1}(\mathbf{T})$.
For each limit ordinal $\lambda \in \mathsf{spec}_{\Sigma_1}(\mathbf{T})$ with $\lambda > \max(\omega, \alpha)$, define

$$A_\lambda := \{a \in L_\lambda \mid \text{there is a } \Sigma_1\text{-definition of } a \text{ in } L_\lambda \text{ using parameters} < \alpha.\}$$

Let ρ_λ be the least ordinal not in A_λ. By the proof of Theorem 7.8. in chap. V of [2], we get

$$A_\lambda = L_{\rho_\lambda} \quad \text{and} \quad \rho_\lambda \text{ is the least } \lambda\text{-stable ordinal} \geq \alpha. \tag{15}$$

In actuality, this proof assumes that λ is admissible. However, ruminating on the proof, it turns out that all which is required is that the predicate $x \in L_\delta$ (of x and δ) and the constructible ordering $<_L$ are absolute for L_λ. Therefore it suffices to assume that λ is a limit $> \omega$ (for more details see [8], II, Theorem 5.2).
Note that $\alpha \leq \rho_\lambda \leq \lambda$.

Case 1: $\alpha < \rho_\lambda$ for some $\lambda \in \mathsf{spec}_{\Sigma_1}(\mathbf{T})$, where λ is a limit $> \max(\omega, \alpha)$.
Then α has a Σ_1-definition in L_λ using parameters $\beta_1, \ldots, \beta_n < \alpha$. Let $\psi(x, \beta_1, \ldots, \beta_n)$ be the defining formula. Put

$$\begin{aligned}\theta(x) :=& \big[\, L_\lambda \models \exists! \xi\, \psi(\xi, \beta_1, \ldots, \beta_n) \;\wedge\; x \in L_\lambda \;\wedge\; L_\lambda \models \psi(x, \beta_1, \ldots, \beta_n)\big] \vee \\ & \big[\, L_\lambda \models \neg \exists! \xi\, \psi(\xi, \beta_1, \ldots, \beta_n) \;\wedge\; x = \lambda\big].\end{aligned} \tag{\dagger}$$

θ is a Σ_1-formula with parameters $\lambda, \beta_1, \ldots, \beta_n$ all of which are in $\mathsf{spec}_{\Sigma_1}(\mathbf{T})$. Using the formulae $\phi_\lambda, \phi_{\beta_1}, \ldots, \phi_{\beta_n}$ (see (14)), we can rewrite θ to an equivalent Σ_1-formula such that $\mathbf{T} \vdash \exists! \eta\, \theta(\eta)$. Moreover, $L \models \theta[\alpha]$. Therefore $\alpha \in \mathsf{spec}_{\Sigma_1}(\mathbf{T})$.

Case 2: For all limits $\lambda \in \mathsf{spec}_{\Sigma_1}(\mathbf{T})$ with $\lambda > \max(\alpha, \omega)$, it holds $\alpha = \rho_\lambda$.
By (15), α is λ-stable for all such λ. Therefore α is $\sigma_{\mathbf{T}}$-stable, as $\sigma_{\mathbf{T}}$ is the limit of all these ordinals. However, from $\mathbf{T} \vdash \exists! \xi\, \psi(\xi)$ with ψ Σ_1, we get $L_{\sigma_{\mathbf{T}}} \models \exists \xi\, \psi(\xi)$ and hence $L_\alpha \models \exists \xi \psi(\xi)$, which yields $\mathsf{spec}_{\Sigma_1}(\mathbf{T}) \subseteq \alpha$, contradicting $\alpha < \sigma_{\mathbf{T}}$. □

A quick glance at the preceding proof reveals that the proof utilizes that T is a classical theory at every turn. Therefore one might expect a different behaviour for intuitionistic set theories. Up till now, however, no intuitionistic set theory int-T has been found where $\mathsf{spec}_{\Sigma_1}(\text{int-}T)$ contains holes. On the other hand, for several intuitionistic theories it has been shown that their spectrum yields a segment like in the classical case. If T is a classical set theory let int-T be the theory with the same axioms but based on intuitionistic logic.

Theorem 6.3 *If* $\mathbf{T}$ *denotes any of the theories* $\mathbf{KPl}$, $\mathbf{KPl}^r$, *or* $\mathbf{KPi}^w$, *then*

$$\mathsf{spec}_{\Sigma_1}(\mathbf{T}) = \mathsf{spec}_{\Sigma_1}(\mathit{int}\text{-}\mathbf{T}).$$

Proof: This has been shown by K. Thiel. The results will be presented in his forthcoming PhD thesis. □

7 Recursively large ordinals and ordinal representation systems

It is probably not widely known that an ordinal analysis of a theory T not only characterises the provably recursive ordinals of T but also provides information about provable sets of higher complexity. For instance the first T-stable ordinal, σ_T, that is the first ordinal ρ such that all Σ_1 definable ordinals of T are $< \rho$ can be obtained from an ordinal analysis of T as well. σ_T can also be characterised as the first ordinal which is closed under all ∞-functions on ordinals which are provably total in T (cf. [12], VIII), or in the case of subsystems of second order arithmetic as the supremum of its provable Δ^1_2 ordinals. To illustrate these more subtle features by means of a simple example (which nevertheless encapsulates the generic case) the last section of this paper introduces an ordinal representation system that has been employed in the ordinal analysis of the subsystem of second order arithmetic, based on Δ^1_2 comprehension and bar induction or equivalently the set theory **KPi**..

7.1 Ordinal functions based on a weakly inaccessible cardinal

Recall that **KPi** is a set theory which originates from Kripke-Platek set theory and in addition has an axiom which says that any set is contained in an admissible set. Thus the standard models of **KPi** in L are the segments L_κ with κ being recursively inaccessible. The ordinal analysis for **KPi** (cf. [14]) used an ordinal representation system built from ordinal functions, so-called *collapsing functions*, which have originally been defined with the help of a weakly inaccessible cardinal. This subsection expounds on the development of this particular ordinal representation system with an eye towards the role of cardinals therein. Let

$$\mathbf{I} \;:=\; \text{“first weakly inaccessible cardinal”} \tag{16}$$

that enumerates the cardinals below $\mathbf{I}$. Further let

$$\Re^{\mathbf{I}} \;:=\; \{\mathbf{I}\} \cup \{\Omega_{\xi+1} : \xi < \mathbf{I}\}. \tag{17}$$

Variables κ, π will range over $\Re^{\mathbf{I}}$.

Definition 7.1 An ordinal representation system for the analysis of **KPi** can be derived from the following functions and Skolem hulls of ordinals defined by recursion on α:

$$C^{\mathbf{I}}(\alpha,\beta) = \begin{cases} \textit{closure of } \beta \cup \{0, \mathbf{I}\} \\ \textit{under:} \\ +, (\xi \mapsto \omega^{\xi}) \\ (\xi \mapsto \Omega_{\xi})_{\xi<\mathbf{I}} \\ (\xi\pi \longmapsto \psi_{\pi}(\xi))_{\xi<\alpha} \end{cases} \tag{18}$$

$$\psi_{\pi}(\alpha) \simeq \min\{\rho < \pi : C^{\mathbf{I}}(\alpha,\rho) \cap \pi = \rho \wedge \pi \in C^{\mathbf{I}}(\alpha,\rho)\}. \tag{19}$$

Note that if $\rho = \psi_{\pi}(\alpha)$, then $\psi_{\pi}(\alpha) < \pi$ and $[\rho,\pi) \cap C^{\mathbf{I}}(\alpha,\rho) = \emptyset$, thus the order-type of the ordinals below π which belong to the Skolem hull $C^{\mathbf{I}}(\alpha,\rho)$ is ρ . In more pictorial terms, ρ is the α^{th} *collapse* of π.

Lemma 7.2 *If $\pi \in C^{\mathbf{I}}(\alpha,\pi)$, then $\psi_{\pi}(\alpha)$ is defined; in particular $\psi_{\pi}(\alpha) < \pi$.*

Proof: Note first that for a limit ordinal λ,

$$C^{\mathbf{I}}(\alpha,\lambda) = \bigcup_{\xi<\lambda} C^{\mathbf{I}}(\alpha,\xi)$$

since the right hand side is easily shown to be closed under the clauses that define $C^{\mathbf{I}}(\alpha,\lambda)$. Thus we can pick $\omega \leq \eta < \pi$ such that $\pi \in C^{\mathbf{I}}(\alpha,\eta)$. Now define

$$\begin{aligned} \eta_0 &= \sup C^{\mathbf{I}}(\alpha,\eta) \cap \pi \\ \eta_{n+1} &= \sup C^{\mathbf{I}}(\alpha,\eta_n) \cap \pi \\ \eta^* &= \sup_{n<\omega} \eta_n. \end{aligned} \tag{20}$$

Since the cardinality of $C^{\mathbf{I}}(\alpha,\eta)$ is the same as that of η and therefore less than π, the regularity of π implies that $\eta_0 < \pi$. By repetition of this argument one obtains $\eta_n < \pi$, and consequently $\eta^* < \pi$. The definition of η^* then ensures

$$C^{\mathbf{I}}(\alpha,\eta^*) \cap \pi = \bigcup_{n} C^{\mathbf{I}}(\alpha,\eta_n) \cap \pi = \eta^* < \pi.$$

Therefore, $\psi_{\pi}(\alpha) < \pi$. □

Let $\varepsilon_{\mathbf{I}+1}$ be the least ordinal $\alpha > \mathbf{I}$ such that $\omega^{\alpha} = \alpha$. The next definition singles out a subset $\mathcal{T}(\mathbf{I})$ of $C^{\mathbf{I}}(\varepsilon_{\mathbf{I}+1}, 0)$ which gives rise to an ordinal representation system, i.e., there is an elementary ordinal representation system $\langle \mathcal{OR}, \lhd, \hat{\Re}, \hat{\psi}, \ldots \rangle$, so that

$$\langle \mathcal{T}(\mathbf{I}), <, \Re, \psi, \ldots \rangle \cong \langle \mathcal{OR}, \lhd, \hat{\Re}, \hat{\psi}, \ldots \rangle. \tag{21}$$

"…" is supposed to indicate that more structure carries over to the ordinal representation system.

Definition 7.3 $\mathcal{T}(\mathbf{I})$ is defined inductively as follows:

1. $0, \mathbf{I} \in \mathcal{T}(\mathbf{I})$.

2. If $\alpha_1, \ldots, \alpha_n \in \mathcal{T}(\mathbf{I})$ and $\alpha_1 \geq \ldots \geq \alpha_n$, then $\omega^{\alpha_1} + \cdots + \omega^{\alpha_n} \in \mathcal{T}(\mathbf{I})$.

3. If $\alpha \in \mathcal{T}(\mathbf{I})$, $0 < \alpha < \mathbf{I}$ and $\alpha < \Omega_\alpha$, then $\Omega_\alpha \in \mathcal{T}(\mathbf{I})$.

4. If $\alpha, \pi \in \mathcal{T}(\mathbf{I})$, $\pi \in C^{\mathbf{I}}(\alpha, \pi)$ and $\alpha \in C^{\mathbf{I}}(\alpha, \psi_\pi(\alpha))$, then $\psi_\pi(\alpha) \in \mathcal{T}(\mathbf{I})$.

The side conditions in 7.3.2, 7.3.3 are easily explained by the desire to have unique representations in $\mathcal{T}(\mathbf{I})$. The requirement $\alpha \in C^{\mathbf{I}}(\alpha, \psi_\pi(\alpha))$ in 7.3.4 also serves the purpose of unique representations (and more) but is probably a bit harder to explain. The idea here is that from $\psi_\pi(\alpha)$ one should be able to retrieve the stage (namely α) where it was generated. This is reflected by $\alpha \in C^{\mathbf{I}}(\alpha, \psi_\pi(\alpha))$.

It can be shown that the foregoing definition of $\mathcal{T}(\mathbf{I})$ is deterministic, that is to say every ordinal in $\mathcal{T}(\mathbf{I})$ is generated by the inductive clauses of 7.3 in exactly one way. As a result, every $\gamma \in \mathcal{T}(\mathbf{I})$ has a unique representation in terms of symbols for $0, \mathbf{I}$ and function symbols for $+, (\alpha \mapsto \Omega_\alpha), (\alpha, \pi \mapsto \psi_\pi(\alpha)$. Thus, by taking some primitive recursive (injective) coding function $\lceil \cdots \rceil$ on finite sequences of natural numbers, we can code $\mathcal{T}(\mathbf{I})$ as a set of natural numbers as follows:

$$\ell(\alpha) = \begin{cases} \lceil 0, 0 \rceil & \text{if } \alpha = 0 \\ \lceil 1, 0 \rceil & \text{if } \alpha = \mathbf{I} \\ \lceil 2, \ell(\alpha_1), \cdots, \ell(\alpha_n) \rceil & \text{if } \alpha = \omega^{\alpha_1} + \cdots + \omega^{\alpha_n} \\ \lceil 3, \ell(\beta) \rceil & \text{if } \alpha = \Omega_\beta \\ \lceil 4, \ell(\beta), \ell(\pi) \rceil & \text{if } \alpha = \psi_\pi(\beta), \end{cases}$$

where the distinction by cases refers to the unique representation of 7.3. With the aid of ℓ, the ordinal representation system of (21) can be defined by letting $\mathcal{OR}$ be the image of ℓ and setting $\lhd \ := \{(\ell(\gamma), \ell(\delta)) : \gamma < \delta \wedge \delta, \gamma \in \mathcal{T}(\mathbf{I})\}$ etc. However, for a proof that this definition of $\langle \mathcal{OR}, \lhd, \hat{\mathfrak{R}}, \hat{\psi}, \ldots \rangle$ in point of fact furnishes an elementary ordinal representation system, we have to refer to the literature (cf. [3, 5, 23]).

7.1.1 Recursively large ordinals

The large cardinal hypothesis that $\mathbf{I}$ is the first weakly inaccessible cardinal is outrageous when compared with the strength of **KPi**. However, it enters the definition procedure of the collapsing function $\psi_{\mathbf{I}}$, which is then employed in the shape of terms to "name" a countable set of ordinals. As one succeeds in establishing recursion relations for the ordering between those terms, the set of terms gives rise to an ordinal representation system. It has long been suggested that, instead, one should be able to interpret the collapsing functions as operating directly on the recursively large counterparts of those cardinals. To give an example, the ordinal notations used in the determination of the ordinals $|\mathbf{ID}_n|$ for theories of n–iterated inductive definitions (cf. [4]) embody collapsing functions $\psi_{\Omega_1}, \ldots, \psi_{\Omega_n}$, which are contingent upon the cardinals $\aleph_1, \ldots, \aleph_n$. The conceptual problem here is that the definition procedure of these functions makes essential use of the set–theoretical universe, whilst the resulting notation system corresponds to a countable, indeed recursive ordinal. Feferman wrote (cf. [9], p. 436):

"It has been suggested that, instead, one should be able to interpret the long hierarchies as operating directly on the (Kripke–Platek) admissible number classes τ_α, where $\tau_1 = \omega_1^{rec}$. However, no theory of such classes currently available allows one to 'name'

higher admissibles in the definition of a function and have a given admissible such as τ_1 closed under it."

For example, taking such an approach in Definition 7.1 would consist in letting

$$\mathbf{I} := \text{first recursively inaccessible ordinal}$$

and setting $\Re^{\mathbf{I}} := \{\pi < \mathbf{I} : \pi \text{ admissible}, \pi > \omega\}$. The difficulties with this approach arise with the proof of Lemma 7.2. One wants to show that, for all α, $\psi_{\mathbf{I}}(\alpha) < \mathbf{I}$. However, the arguments of the cardinal setting no longer work here. To get a similar result for a recursively inaccessible ordinal κ one would have to work solely with κ-recursive operations. In addition, the functions $\psi_\pi : \varepsilon_{\kappa+1} \to \pi$ would have to be defined for admissible ordinals π with $\omega < \pi < \kappa$. In the cardinal setting this comes down to a simple cardinality argument. To get a similar result for an admissible π one would have to work solely with π-recursive operations. How this can be accomplished is far from being clear as the definition of $C^{\mathbf{I}}(\alpha, \rho)$ for $\rho < \pi$ usually refers to higher admissibles than just π. Notwithstanding that, the admissible approach is workable as was shown in [21, 22, 28]. A key idea therein is that the higher admissibles which figure in the definition of $\psi_\pi(\alpha)$ can be mimicked via names within the structure L_π in a π-recursive manner.

The drawback of the admissible approach is that it involves quite horrendous definition procedures and computations, which when taken as the first approach are at the limit of human tolerance.

On the other hand, the admissible approach provides a natural semantics for the terms in the ordinal representation systems. Recalling the notion of *good* Σ_1*-definition* from Definition 6.1, it turns out that all the ordinals of $\mathcal{T}(\mathbf{I}) \cap \mathbf{I}$ possess a good Σ_1-definition in **KPi** (cf. [22]) under the interpretation which takes $\mathbf{I}$ to be the first recursively inaccessible ordinal and lets the functions ψ_π operate on admissible ordinals π instead of regular cardinals.

Unlike in the case of **KP**, $\mathcal{T}(\mathbf{I}) \cap \mathbf{I}$ only forms a proper subset of $\mathsf{spec}_{\Sigma_1}(\mathbf{KPi})$ with many 'holes'.[3] To illuminate the nature of the ordinals in $\mathcal{T}(\mathbf{I}) \cap \mathbf{I}$, it would be desirable to find another property which distinguishes them among the ordinals of $\mathsf{spec}_{\Sigma_1}(\mathbf{KPi})$.

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