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V. STOLTENBERG-HANSEN and J. TUCKER

REPORT No. 41, 2000/2001
ISSN 1103-467X
ISRN IML-R- -41-00/01- -SE

INSTITUT MITTAG-LEFFLER
THE ROYAL SWEDISH ACADEMY OF SCIENCES

Computable and continuous partial homomorphisms on metric partial algebras

Viggo Stoltenberg-Hansen*
Department of Mathematics, Uppsala University, Box 480,
S-75106 Uppsala, Sweden

John V.Tucker*
Department of Computer Science, University of Wales Swansea, Singleton Park,
Swansea, SA2 8PP, Wales

Abstract

We analyse the connection between the computability and continuity of functions in the case of homomorphisms between topological algebraic structures. Inspired by the Pour-El and Richards equivalence theorem between computability and boundedness for closed linear operators on Banach spaces, we study the rather general situation of partial homomorphisms between metric partial universal algebras. First, we develop a set of basic notions and results that reveal some of the delicate algebraic, topological and effective properties of partial algebras. Our main computability concepts are based on numerations and include those of effective metric partial algebras and effective partial homomorphisms. We prove a general equivalence theorem that includes the Pour-El and Richards Theorem, and has other applications. Finally, the Pour-El and Richards axioms for computable sequence structures on Banach spaces are generalised to computable partial sequence structures on metric algebras, and we prove their equivalence with our computability model based on numerations.

1 Introduction

The connection between "computability" and "continuity" has been explored in Intuitionistic and Constructive Analysis, and in its classical alter ego Computable Analysis. Early on constructive real functions on real numbers were shown to be continuous by Brouwer. Ceitin [5] shows that computable

*Partially supported by Swedish Research Council for Engineering Sciences, Project 221-97-745.

functions on constructive complete metric spaces are continuous. The connection between computability and continuity is essential for Computable Analysis.

The computability-continuity connection was fundamental in the study of higher type functionals by Kleene [14] and Kreisel [15] and was exploited in generalised recursion theory and proof theory. It also belongs to the theory of the Myhill-Shepherdson Theorem and the Kreisel-Lacombe-Shoenfield Theorem. Against this background, the computability-continuity connection was a basis for the development of domain theory by Scott and Ershov. One of the main messages of domain theory is that the essential structural features of computability are generalised by continuity. In domain theory continuity is developed from an order-theoretic analysis of approximations and leads to a theory of continuous functions on certain T_0 spaces. Occasionally, over enthusiastic domain theorists have been known to make the identification

$$\text{computability} = \text{continuity}$$

which, of course, is a mistake. Computability has to be imposed on the domains and continuous functions through numberings and there is an appropriate effective domain theory (initially due to Scott and Ershov: see Stoltenberg-Hansen et al. [24]).

Not surprisingly perhaps, Computable Analysis has developed independently from higher type recursion theory and domain theory. Theorems of analytical interest were pursued for specific spaces, using concrete representations, usually made from numberings, to introduce computability.

An early general framework for analysing the theory of these representations was the theory of effective metric spaces (Moschovakis [20]). Connections came later, in the use of cpos (Weihrauach and Schreiber [34]), algebraic domains (Stoltenberg-Hansen and Tucker [25, 26, 27], Blanck [3]) and continuous domains (Edalat; for a survey see [7]) to give an abstract theory of approximation applicable to Computable Analysis. The use of recursive operators on Baire space was used in a systematic theory starting in Weihrauch [32] and culminating in Weihrauch [33]. In Stoltenberg-Hansen and Tucker [29] there is a series of equivalence proofs between these approaches.

In Computable Analysis, Pour El and Richards discovered several exciting theorems, examples, and new methods. Their new approach, presented in full in Pour-El and Richards [23], was to work with very limited assumptions about classical computability via recursive functions and to banish codings as an object of interest. Using simple axioms for computable sequence structures on a Banach space, they proved results about the computability-continuity connection, and for the first time a theorem, roughly of the form (in its positive direction): *for closed linear operators on*

Banach spaces with computable sequence structures and a computable generating sequence "computability = continuity" (First Main Theorem, p. 101). This striking fact covers a rich collection of applications, including their earlier influential study of non-computable solutions to the Wave Equation (Pour-El and Richards [22]), and some recent development in Weihrauch and Zong [35, 36].

However, for students of Computable Analysis, who see representations of spaces as an essential object of interest, their axiomatic approach hides a great deal. In fact some of the Pour-El and Richards' theorems require re-investigation to isolate in greater detail their effective content, and connect the axiomatic approach with the more representation-friendly approaches mentioned earlier. For example, in Stoltenberg-Hansen and Tucker [29] the equivalence between algebraic domains and the Pour-El Richards approach is proved for Banach spaces.

In this paper we analyse the computability-continuity connection in the general setting of homomorphisms of universal algebras equipped with a topology defined by a metric. Furthermore, we extend the usual setting by considering *partial algebras*, that is algebras whose operations may be partial, and *partial homomorphisms* between such algebras. This generalisation is necessary because partial functions play an essential role in computing with topological algebras. The generalisation is also nontrivial, in particular in regard to effectivity, and necessitates a careful analysis of many basic concepts. We analyse thoroughly the general situation of an effective metric partial algebra. We prove the following theorem which establishes in an explicit way the effective content of partial homomorphisms (all unexplained terminology is defined later).

Theorem *Let A and B be metric partial Σ-algebras and let (X, α) and (Y, β) be effective metric partial Σ-subalgebras of A and B, respectively, such that X is generated algebraically by an α-computable partial sequence $e: \mathbb{N} \to X$. Let $(\bar{X}^A_{k,\alpha}, \bar{\alpha})$ and $(\bar{Y}^B_{k,\beta}, \bar{\beta})$ be the the effective metric spaces of computable elements in A and B, respectively.*

Suppose $h: \bar{X}^A_{k,\alpha} \to \bar{Y}^B_{k,\beta}$ is a partial function such that $he(\mathbb{N}) \subseteq Y$, $he: \mathbb{N} \to Y$ is a partial β-computable sequence, and $h|_X: X \to Y$ is a strong partial Σ-homomorphism. Then the following are equivalent.

(i) *h is $(\bar{\alpha}, \bar{\beta})$-computable.*

(ii) h is $(\bar{\alpha}, \bar{\beta})$-*Lacombe continuous and* $\mathrm{dom}(h)$ is $\bar{\alpha}$-*semicomputable.*

(iii) *h is $\bar{\alpha}$-effectively continuous and* $\mathrm{dom}(h)$ is $\bar{\alpha}$-*semicomputable.*

We show how, as the general structure of the metric partial algebra is replaced by more specific algebraic structures, such as topological groups and

Banach spaces, interesting results can be deduced as special cases, including the striking First Main Theorem in Pour El and Richards [23]. These applications concern the relationship between homomorphisms and effective uniform continuity. For topological groups, continuous homomorphisms are uniformly continuous. In the case of linear operators in Banach spaces, which are such homomorphisms, of course, bounded operators are precisely the effectively uniformly continuous operators.

The structure of the paper is as follows. In Section 2 we explain the basic properties of effective partial algebras and partial homomorphisms. In Section 3 we study the theory of effective metric spaces and partial continuous functions without reference to algebra. We give a proof of Ceitin's theorem which is fundamental for our main theorem. In Section 4 we introduce effective metric partial algebras and prove the main theorem stated in the introduction. We also give some applications to Banach spaces and topological groups. Finally, in Section 5 we present our generalisation of the Pour-El and Richards approach and prove an equivalence with effective metric partial algebras.

The prerequisites for this paper are some knowledge of Computable Analysis (Aberth [1], Pour-El and Richards [23], and Weihrauch [33]), the theory of effective metric spaces (Moschovakis [20]), effective and computable numberings (Ershov [8, 9, 10], Stoltenberg-Hansen and Tucker [27, 28]) and some universal algebra (Grätzer [12], Wechler [31], and Meinke and Tucker [18]).

Our recursion theoretic notation is standard. We let $\lambda xy.\langle x, y\rangle$ be a primitive recursive pairing function, i.e., a primitive recursive bijection between $\mathbb{N}\times\mathbb{N}$ and $\mathbb{N}$, extended in the usual way to more arguments. The e^{th} partial recursive function (in some standard enumeration) of n arguments is denoted by $\{e\}^n$, though the n will mostly be suppressed since the arity will be apparent. The e^{th} r.e. set is $W_e = \mathrm{dom}(\{e\}^1)$. We will use a primitive recursive function $\lambda en.W_e^n$ approximating the r.e. sets, where W_e^n is a (canonical index for a) finite set, such that $m \leq n \implies W_e^m \subseteq W_e^n$, and $W_e = \cup_{n\in\omega} W_e^n$.

We will be dealing with partial functions. Suppose $f, g\colon A \to B$ are partial functions and $a \in A$. Then we write

$$f(a) \simeq g(a)$$

if either both $f(a)$ and $g(a)$ are defined and equal or both are undefined. This is the *Kleene* equality or *strong* equality. We write $f(a){\downarrow}$ if $f(a)$ is defined and $f(a){\uparrow}$ otherwise.

Suppose $f\colon A \to B$ and $g\colon B \to C$ are partial functions. Then

$$h(x) \simeq g(f(x))$$

means that $h(x){\downarrow}$ if, and only if, there is $b \in B$ such that $f(x) \simeq b$ and $g(b){\downarrow}$, and then $h(x) \simeq g(b)$.

2 Effective partial algebras and partial homomorphisms

In this section we present some basic notions and results about partial algebras and effective partial algebras. The topological or metric aspects of effective partial algebras are dealt with in later sections.

2.1 Preliminaries on partial algebras

We assume the reader knows the basic theory of universal algebras: we need the concepts of signature Σ, Σ-algebra, Σ-subalgebra, Σ-generating sequence, Σ-term, Σ-term evaluation, Σ-homomorphism etc. Although the theory is intended to work for many sorted universal algebras, we will use single sorted partial algebras in our exposition. We refer to Grätzer [12], Wechler [31] and Meinke and Tucker [18] for background material on algebra.

We are going to consider partial algebras and partial homomorphisms between partial algebras. The literature on universal algebras almost exclusively deals with total algebras and total homomorphisms. There does not seem to be a consensus on what one should mean by a partial homomorphism between partial algebras or even a total homomorphism between partial algebras. For example, Grätzer [12] gives three definitions of total homomorphisms between partial algebras. Therefore we will be precise about our choice of concepts and make some useful elementary observations.

Definition 2.1. Let Σ be a signature. A *partial* Σ*-algebra* A is a Σ-algebra whose operations are partial functions.

Note that if $c \in \Sigma$ is a constant, i.e. a 0-ary operation, then the interpretation c_A of c in A is defined, i.e. an element of A.

Definition 2.2. Let A be a partial Σ-algebra. Then a non-empty set $X \subseteq A$ constitutes a *partial* Σ*-subalgebra* of A if for each k-ary $\sigma \in \Sigma$ and $x_1, \ldots, x_k \in X$,

$$\sigma_A(x_1, \ldots, x_k)\downarrow \Rightarrow \sigma_A(x_1, \ldots, x_k) \in X.$$

We choose the following notion of partial Σ-homomorphism.

Definition 2.3. Let A and B be partial Σ-algebras. A partial map $h\colon A \to B$ is a *partial* Σ*-homomorphism* if for any k-ary $\sigma \in \Sigma$ and $a_1, \ldots, a_k \in A$, if $\sigma_B(h(a_1), \ldots, h(a_k))\downarrow$ then

$$h(\sigma_A(a_1, \ldots, a_k)) \simeq \sigma_B(h(a_1), \ldots, h(a_k)).$$

Note that if B is a total Σ-algebra then h is a partial Σ-homomorphism if for each k-ary $\sigma \in \Sigma$, if $a_1, \ldots a_k \in \mathrm{dom}(h)$ then

$$h(\sigma_A(a_1, \ldots, a_k)) \simeq \sigma_B(h(a_1), \ldots, h(a_k)).$$

Let $C[0,1]$ be the set of continuous functions from the unit interval into the reals $\mathbb{R}$, and consider $C[0,1]$ as a Σ-algebra with addition and a unary operation for each $a \in \mathbb{R}$ corresponding to scalar multiplication. Let

$$D: C[0,1] \to C[0,1]$$

be the differentiation operator. Then D is a partial Σ-homomorphism.

If $h: A \to B$ is a partial Σ-homomorphism then $c_A \in \text{dom}(h)$ for each constant $c \in \Sigma$ and its image $\text{im}(h)$ is a partial Σ-subalgebra of B, but its domain $\text{dom}(h)$ need not be a partial Σ-subalgebra of A. However, if B is a total Σ-algebra then $\text{dom}(h)$ is a total Σ-subalgebra of A.

Let $V = \{v_i : i \in \mathbb{N}\}$ be a set of distinct variables. Let $T(\Sigma, V)$ be the Σ-term algebra with variables from V; this is a *total* Σ-algebra.

By an easy induction on terms we obtain

Lemma 2.4. *Let A and B be partial Σ-algebras and let $h: A \to B$ be a partial Σ-homomorphism. Then for any $t(v_1, \ldots, v_k) \in T(\Sigma, V)$ and $a_1, \ldots, a_k \in A$, if $t^B(h(a_1), \ldots, h(a_k))\downarrow$ then*

$$h(t^A(a_1, \ldots, a_k)) \simeq t^B(h(a_1), \ldots, h(a_k)).$$

In our main theorem we need a stronger form of partial homomorphism working on certain subalgebras.

Definition 2.5. Let A and B be partial Σ-algebras. A partial map $h: A \to B$ is a *strong partial Σ-homomorphism* if for any k-ary $\sigma \in \Sigma$ and $a_1, \ldots, a_k \in A$

$$h(\sigma_A(a_1, \ldots, a_k)) \simeq \sigma_B(h(a_1), \ldots, h(a_k)).$$

In case h is total the above notion is that of strong Σ-homomorphism in Grätzer [12].

Lemma 2.6. *Let A and B be partial Σ-algebras and let $h: A \to B$ be a strong partial Σ-homomorphism. Then for any $t(v_1, \ldots, v_k) \in T(\Sigma, V)$ and $a_1, \ldots, a_k \in A$*

$$h(t^A(a_1, \ldots, a_k)) \simeq t^B(h(a_1), \ldots, h(a_k)).$$

Proof. By induction on terms. The lemma is true for constants and variables. Suppose $t(\vec{v}) = \sigma(t_1(\vec{v}), \ldots, t_s(\vec{v}))$, where $s \geq 1$. Then inductively, for $\vec{a} \in A$ and $1 \leq i \leq s$, $h(t_i^A(\vec{a})) \simeq t_i^B(h(\vec{a}))$. We claim

$$h(\sigma_A(t_1^A(\vec{a}), \ldots, t_s^A(\vec{a}))) \simeq \sigma_B(h(t_1^A(\vec{a})), \ldots, h(t_s^A(\vec{a}))). \qquad (1)$$

Suppose $t_i^A(\vec{a})\downarrow$ for each i. Then (1) holds since h is a strong Σ-homomorphism. Now suppose $t_i^A(\vec{a})\uparrow$ for some i. Then $h(t_i^A(\vec{a}))\uparrow$ and hence both sides of (1) are undefined. Thus (1) holds also in this case. Combining this with the induction hypothesis we obtain

$$\begin{aligned} h(t^A(\vec{a})) &\simeq h(\sigma_A(t_1^A(\vec{a}),\ldots,t_s^A(\vec{a}))) \\ &\simeq \sigma_B(t_1^B(h(\vec{a})),\ldots,t_s^B(h(\vec{a}))) \\ &\simeq t^B(h(\vec{a})). \end{aligned}$$

□

Let A be a partial Σ-algebra and let $e\colon \mathbb{N} \to A$ be a partial sequence. We define the *term evaluation map* $\mathrm{TE}_e\colon T(\Sigma, V) \to A$ inductively as follows:

$\mathrm{TE}_e(c) \simeq c_A$	$c \in \Sigma$ constant
$\mathrm{TE}_e(v_i) \simeq e(i)$	$v_i \in V$
$\mathrm{TE}_e(\sigma(t_1,\ldots,t_k)) \simeq \sigma_A(\mathrm{TE}_e(t_1),\ldots,\mathrm{TE}_e(t_k))$	$\sigma \in \Sigma$ k-ary.

From its definition we see that TE_e is a strong partial Σ-homomorphism. Note that $\mathrm{dom}(\mathrm{TE}_e)$ need not be a partial Σ-subalgebra but the image $\mathrm{im}(\mathrm{TE}_e)$ is a partial Σ-subalgebra, the partial Σ-subalgebra generated by the elements of the partial sequence e.

Notation: The Σ-subalgebra $\mathrm{im}(\mathrm{TE}_e)$, the Σ-subalgebra generated by e, is denoted by $\langle e \rangle$.

Lemma 2.7. *Let A be a partial Σ-algebra and B a total Σ-algebra and let $e\colon \mathbb{N} \to A$ be a partial sequence. If $h\colon A \to B$ is a partial Σ-homomorphism such that* $\mathrm{im}(e) \subseteq \mathrm{dom}(h)$ *then $h|_{\langle e \rangle}\colon \langle e \rangle \to B$ is a strong total Σ-homomorphism.*

Proof. It suffices to show that $\langle e \rangle \subseteq \mathrm{dom}(h)$. Suppose $x \in \langle e \rangle$. Thus $x = t^A(e(i_1),\ldots,e(i_k))$ for some term t where $e(i_j)\downarrow$ for $j = 1,\ldots,k$. Then $h(x)\downarrow$ by Lemma 2.4. □

2.2 Effective partial algebras

First we define the notion of a numbered set, effective subsets of a numbered set, and effective partial functions between numbered sets.

Definition 2.8. A *numbered set* is a set X together with a numbering α, i.e., a surjective total function $\alpha\colon \Omega_\alpha \to X$, where $\Omega_\alpha \subseteq \mathbb{N}$. The set X with the numbering α is denoted by (X, α).

Definition 2.9. Let (X, α) be a numbered set.

(i) A subset S of X is *α-semicomputable* if there is an r.e. set $W \subseteq \mathbb{N}$ such that

$$\alpha^{-1}(S) = \Omega_\alpha \cap W.$$

(ii) A subset S of X is *α-computably enumerable* if there is an r.e. set V such that $V \subseteq \Omega_\alpha$ and $\alpha(V) = S$.

Equivalently, (i) is simply: the set $\alpha^{-1}(S)$ is r.e. in Ω_α. This notion is called *listable* in Moschovakis [21].

Definition 2.10. Let (X, α) and (Y, β) be numbered sets and let $f\colon X \to Y$ be a partial function.

(i) f is *(α, β)-computable* if there is a partial recursive function $\bar{f}\colon \mathbb{N} \to \mathbb{N}$ such that for each $n \in \Omega_\alpha$,

 (a) $\bar{f}(n)\downarrow \implies \bar{f}(n) \in \Omega_\beta$; and

 (b) $f(\alpha(n)) \simeq \beta(\bar{f}(n))$.

(ii) f is *weakly (α, β)-computable* if there is a partial recursive function $\bar{f}\colon \mathbb{N} \to \mathbb{N}$ such that

$$f(\alpha(n))\downarrow \implies f(\alpha(n)) \simeq \beta(\bar{f}(n)).$$

For a partial sequence $e\colon \mathbb{N} \to X$ we say e is (weakly) α-computable for e being (weakly) (id, α)-computable, i.e. $\mathbb{N}$ is numbered by the identity function.

Note that if $S \subseteq X$ and f is (weakly) (α, β)-computable then $f|_S\colon S \to Y$ is (weakly) (α', β)-computable where $\Omega_{\alpha'} = \alpha^{-1}(S)$ and α' is the restriction of α to $\Omega_{\alpha'}$.

Lemma 2.11. *Let (X, α) and (Y, β) be numbered sets. A set $S \subseteq X$ is α-semicomputable if, and only if, there is an (α, β)-computable partial function $f\colon X \to Y$ such that* $\mathrm{dom}(f) = S$.

Proof. Suppose $f\colon X \to Y$ is a partial (α, β)-computable function tracked by a partial recursive function $\bar{f}$. Then it follows that

$$\alpha^{-1}(\mathrm{dom}(f)) = \Omega_\alpha \cap \mathrm{dom}(\bar{f}).$$

Conversely, suppose $S \subseteq X$ is α-semicomputable. Let W be an r.e. set such that $\alpha^{-1}(S) = \Omega_\alpha \cap W$. Let $n_0 \in \Omega_\beta$ and define a partial function $f\colon X \to Y$ by $f(x) \simeq \beta(n_0) \iff x \in S$. Then f is (α, β)-computable, tracked by the partial recursive function $\bar{f}(n) \simeq n_0 \iff n \in W$. □

In fact the two concepts of computability are connected by the semicomputability of the domain.

Lemma 2.12. *Let (X, α) and (Y, β) be numbered sets and let $f\colon X \to Y$ be a partial function. Then f is (α, β)-computable if, and only if, f is weakly (α, β)-computable and* $\mathrm{dom}(f)$ *is α-semicomputable.*

Proof. For the if direction let W be an r.e. set witnessing that $\mathrm{dom}(f)$ is α-semicomputable. Let $\bar{f}$ be a partial recursive tracking function for f. We may assume $\bar{f}(n)\downarrow \implies n \in W$. Suppose $n \in \Omega_\alpha$. If $\bar{f}(n)\downarrow$ then $n \in W$ so $f(\alpha(n))\downarrow$ and hence $f(\alpha(n)) \simeq \beta(\bar{f}(n))$. In particular $\bar{f}(n) \in \Omega_\beta$. Furthermore, if $f(\alpha(n))\uparrow$ then $n \notin W$ so $\bar{f}(n)\uparrow$ and (b) holds. The converse is similar. □

Thus for total functions computable and weakly computable coincide.

We now define the notion of an effective partial Σ-algebra by requiring the partial operations to be effective.

Definition 2.13. Let X be a partial Σ-algebra and let α be a numbering of X.

(i) (X, α) is an *effective* partial Σ-algebra if each k-ary partial operation σ of X is (α^k, α)-computable, where α^k is the product numbering of X^k obtained from α.

(ii) (X, α) is a *weakly effective* partial Σ-algebra if each k-ary partial operation σ of X is weakly (α^k, α)-computable.

Normally Σ is finite. However we also allow Σ to be infinite in which case we require a recursive enumeration σ_i of the operations in Σ along with a recursive enumeration of their arities k_i. Then we require in Definition 2.13 that a tracking function for σ_i is obtained uniformly from i. A typical example is a vector space over, say, the field of rational numbers $\mathbb{Q}$, where each $q \in \mathbb{Q}$ determine a unary operation on the abelian group of vectors, viz., scalar multiplication. All our arguments in the sequel are uniform in Σ.

Notice that the code set Ω_α equipped with the restrictions of the partial recursive tracking functions $\bar{\sigma}$ to Ω_α form a partial Σ-algebra of numbers. If (X, α) is a weakly effective partial Σ-algebra then $\alpha: \Omega_\alpha \to X$ is a (total) Σ-homomorphism. If (X, α) is an effective partial Σ-algebra then α is a strong Σ-homomorphism.

Recursive real numbers

Recall the definition of the set $\mathbb{R}_k$ of recursive real numbers in terms of computable Cauchy sequences of rational numbers (for example, see Pour-El and Richards [23]). Each recursive real is represented by codes for a recursive sequence of rationals and a recursive modulus of convergence. Such a coding is a standard numbering $\rho: \Omega_\rho \to \mathbb{R}_k$. $\mathbb{R}_k$ is a subfield of the field $\mathbb{R}$ of real numbers and the field operations, including the partial operation of inverse, can be recursively tracked in the coding defined by ρ. In particular, for the partial recursive tracking function $\bar{f}$ of the inverse we have for any $n \in \Omega_\rho$,

$$\bar{f}(n)\downarrow \iff \rho(n) \neq 0.$$

Proposition 2.14. $(\mathbb{R}_k, \rho)$ *is an effective partial algebra, a field. Furthermore the relation* $<$ *is* ρ*-semicomputable.*

It is a fact that the set $\{0\}$ is not ρ-semicomputable. Consider the partial function $f: \mathbb{R}_k \to \mathbb{R}_k$ taking 0 to 0 and being undefined elsewhere. Then f is an example of a weakly (ρ, ρ)-computable function which is not (ρ, ρ)-computable.

Notation In order to simplify notation we will not syntactically distinguish between, e.g., $+$ on $\mathbb{R}$ and a tracking function for $+$ with respect to ρ. Similarly $<$ will denote the less than relation on $\mathbb{R}$ or its r.e. tracking relation. Often we also write a natural or rational number in its usual notation and mean its ρ-index. What is meant is easily seen from the context.

Standard numberings

Let γ be a *standard* computable numbering or encoding of the total term algebra $T(\Sigma, V)$, where $V = \{v_i : i \in \mathbb{N}\}$ (see [27]). We will often follow the common convention that $\ulcorner t \urcorner$ is the γ-index for the term $t \in T(\Sigma, V)$. The standard numbering ensures that $\ulcorner t \urcorner$ is effectively found from t.

Let X be a partial Σ-algebra. For a partial sequence $e: \mathbb{N} \to X$ we let $\mathrm{TE}_e: T(\Sigma, V) \to X$ be the corresponding term evaluation map. Then we define the partial function $\gamma_e: \mathbb{N} \to X$ by

$$\gamma_e(n) \simeq \mathrm{TE}_e(\gamma(n)).$$

Letting $\Omega_e = \mathrm{dom}(\gamma_e)$ we obtain $\gamma_e: \Omega_e \to \langle e \rangle$, the *standard numbering* of the partial Σ-subalgebra generated by e. By an easy induction on terms we see that

$$\gamma_e(\ulcorner t(v_{i_1}, \dots v_{i_k}) \urcorner) \simeq t^X(e(i_1), \dots e(i_k)).$$

Clearly $(\langle e \rangle, \gamma_e)$ is a numbered set.

Lemma 2.15. *A partial sequence* $e: \mathbb{N} \to \langle e \rangle$ *is weakly* γ_e*-computable and* $\mathrm{TE}_e: T(\Sigma, V) \to \langle e \rangle$ *is weakly* (γ, γ_e)*-computable.*

Proof. A tracking function $\bar{e}$ for e is defined by $\bar{e}(i) = \ulcorner v_i \urcorner$. The identity function tracks TE_e. □

Note that we thus far have made no effectivity requirement on e. Nonetheless we have

Lemma 2.16. $(\langle e \rangle, \gamma_e)$ *is a weakly effective partial* Σ*-algebra.*

Proof. Let $\sigma \in \Sigma$ be an k-ary operation for $k \geq 1$, and suppose

$$\sigma_X(\gamma_e(\ulcorner t_1 \urcorner), \dots, \gamma_e(\ulcorner t_k \urcorner))\downarrow.$$

Recall that $\gamma_e(\ulcorner t \urcorner) \simeq \mathrm{TE}_e(t)$. By the homomorphism property,

$$\sigma_X(\mathrm{TE}_e(t_1), \ldots \mathrm{TE}_e(t_k)) \simeq \mathrm{TE}_e(\sigma(t_1, \ldots, t_k)) \simeq \gamma_e(\ulcorner \sigma(t_1, \ldots, t_k) \urcorner).$$

Let $\bar{\sigma}$ be the recursive tracking function for σ on $T(\Sigma, V)$ with respect to γ, i.e., $\bar{\sigma}(\ulcorner t_1 \urcorner, \ldots, \ulcorner t_k \urcorner) = \ulcorner \sigma(t_1, \ldots, t_k) \urcorner$. Then $\bar{\sigma}$ tracks σ showing that σ_X is weakly (γ_e^k, γ_e)-computable. □

By Lemma 2.12, $(\langle e \rangle, \gamma_e)$ is effective if, and only if, $\mathrm{dom}(\sigma_X|_{\langle e \rangle})$ is γ_e-semicomputable for each $\sigma \in \Sigma$. In particular, if X is a total Σ-algebra then $(\langle e \rangle, \gamma_e)$ is an effective Σ-algebra.

Similarly, the strong partial Σ-homomorphism TE_e is (γ, γ_e)-computable if, and only if, Ω_e is r.e. In fact, Ω_e being r.e. settles matters for γ_e.

Proposition 2.17. *Let X be a partial Σ-algebra and let $e\colon \mathbb{N} \to X$ be a partial sequence. Then Ω_e is r.e. if, and only if, e is γ_e-computable and $(\langle e \rangle, \gamma_e)$ is effective.*

Proof. Note that e is γ_e-computable if, and only if, the set $\{i : e(i)\downarrow\}$ is r.e. To prove the if direction let W_σ be an r.e. relation witnessing that $\mathrm{dom}(\sigma)$ is γ_e-semicomputable for $\sigma \in \Sigma$. Define by the second recursion theorem an r.e. set W by

$$\begin{aligned} n \in W \iff & (n = \ulcorner t \urcorner \in \Omega_\gamma) \ \wedge \\ & [(t = c, \ c \in \Sigma \text{ a constant symbol}) \\ & \vee (t = v_i \wedge e(i)\downarrow) \\ & \vee (t = \sigma(t_1, \ldots, t_k) \wedge \forall i. \ulcorner t_i \urcorner \in W \ \wedge \\ & \ (\ulcorner t_1 \urcorner, \ldots, \ulcorner t_k \urcorner) \in W_\sigma; \ \sigma \in \Sigma, \ k\text{-ary})]. \end{aligned}$$

We show $\ulcorner t \urcorner \in W \implies \mathrm{TE}_e(t)\downarrow$ by induction on terms t. The cases $t = c$ or $t = v_i$ are trivial so consider the case $t = \sigma(t_1, \ldots, t_k) \in W$. Then $\ulcorner t_i \urcorner \in W$ for each i, so, inductively, $\mathrm{TE}_e(t_i)\downarrow$. Now

$$\begin{aligned} \mathrm{TE}_e(\sigma(t_1, \ldots, t_k)) &\simeq \sigma_X(\mathrm{TE}_e(t_1), \ldots, \mathrm{TE}_e(t_k)) \\ &\simeq \sigma_X(\gamma_e(\ulcorner t_1 \urcorner), \ldots, \gamma_e(\ulcorner t_k \urcorner)) \end{aligned}$$

and the latter is defined since $(\ulcorner t_1 \urcorner, \ldots, \ulcorner t_k \urcorner) \in W_\sigma$. The converse implication is similar, thus showing $W = \Omega_e$.

For the only if direction suppose Ω_e is r.e. Then

$$e(i)\downarrow \iff \mathrm{TE}_e(v_i)\downarrow \iff \ulcorner v_i \urcorner \in \Omega_e$$

so e is γ_e-computable. Let $\sigma \in \Sigma$ be k-ary. Then, by the strong homomorphism property of TE_e

$$W_\sigma = \{(\ulcorner t_1 \urcorner, \ldots, \ulcorner t_k \urcorner) : \ulcorner \sigma(t_1, \ldots, t_k) \urcorner \in \Omega_e\}$$

is an r.e. relation witnessing that σ is (γ_e^k, γ_e)-computable. □

Lemma 2.18. *Let $h: X \to Y$ be a strong partial homomorphism between the partial Σ-algebras X and Y and let $e: \mathbb{N} \to X$ be a partial sequence. Then $h|_{\langle e\rangle}: \langle e\rangle \to \langle he\rangle$ is weakly (γ_e, γ_{he})-computable.*

Proof. Note that for a Σ-term $t(v_{i_1}, \ldots, v_{i_k})$,

$$\gamma_e(\ulcorner t(v_{i_1}, \ldots, v_{i_k})\urcorner) \simeq t^X(e(i_1), \ldots, e(i_k))$$

and similarly for γ_{he} and Y. We claim that the identity tracks h. For suppose $h(\gamma_e(\ulcorner t(v_{i_1}, \ldots, v_{i_k})\urcorner))\!\downarrow$. Then $e(i_j)\!\downarrow$ for each $j = 1, \ldots, k$ and hence, by Lemma 2.6,

$$\begin{aligned} h(t^X(e(i_1), \ldots, e(i_k))) &\simeq t^Y(he(i_1), \ldots, he(i_k)) \\ &\simeq \gamma_{he}(\ulcorner t(v_{i_1}, \ldots, v_{i_k})\urcorner). \end{aligned}$$

This shows that h is weakly (γ_e, γ_{he})-computable. □

We finally consider the case when we are given an effective partial Σ-algebra (X, α).

Theorem 2.19. *Let (X, α) be an effective partial Σ-algebra and let $e: \mathbb{N} \to X$ be an α-computable partial sequence. Then the following hold.*

(i) *TE_e is (γ, α)-computable.*

(ii) *Ω_e is r.e.*

(iii) *$(\langle e\rangle, \gamma_e)$ is an effective partial Σ-algebra.*

(iv) *The inclusion $\iota: \langle e\rangle \to X$ is (γ_e, α)-computable.*

Proof. For each constant symbol $c \in \Sigma$ let $\bar{c}$ be an α-index for c_X. For each k-ary operation symbol $\sigma \in \Sigma$ let $\bar{\sigma}$ be a partial recursive function witnessing that σ_X is (α^k, α)-computable. Finally let $\bar{e}$ be a partial recursive tracking function for the α-computable partial sequence e. To prove (i) define via the second recursion theorem a partial recursive function f by

$$f(\ulcorner t\urcorner) \simeq \begin{cases} \bar{c} & \text{if } t = c \quad c \text{ a constant} \\ \bar{e}(i) & \text{if } t = v_i \\ \bar{\sigma}(f(\ulcorner t_1\urcorner), \ldots, f(\ulcorner t_k\urcorner)) & \text{if } t = \sigma(t_1, \ldots, t_k) \quad \sigma\ k\text{-ary}. \end{cases}$$

It is straight forward, by induction on terms t, to show that f is a tracking function for TE_e being (γ, α)-computable. It follows from Lemma 2.12 that Ω_e is r.e. and hence, by Proposition 2.17, that $(\langle e\rangle, \gamma_e)$ is effective. The inclusion mapping $\iota: \langle e\rangle \to X$ is also tracked by f. □

We close this section by considering invariance via reductions and equivalences between numberings.

Definition 2.20. Let X be a set numbered by α and β. Then α *recursively reduces to* β, denoted $\alpha \leq \beta$, if there is a partial recursive function $g: \mathbb{N} \to \mathbb{N}$ such that for $n \in \Omega_\alpha$ we have $\alpha(n) \simeq \beta(g(n))$. Furthermore, α is *recursively equivalent to* β, denoted $\alpha \sim \beta$, if α recursively reduces to β and β recursively reduces to α.

The following is straight forward to prove.

Lemma 2.21. *Let X be numbered by α and α' and let Y be numbered by β and β'. Assume $\alpha \leq \alpha'$ and $\beta' \leq \beta$.*

(i) *If $S \subseteq X$ is α'-semicomputable then S is α-semicomputable.*

(ii) *If $f: X \to Y$ is a (weakly) (α', β')-computable partial function then f is (weakly) (α, β)-computable.*

Lemma 2.22. *Let (X, α) be an effective partial Σ-algebra and let $e: \mathbb{N} \to X$ be a partial sequence. Then e is α-computable if, and only if, e is γ_e-computable and $\gamma_e \leq \alpha$.*

Proof. Suppose e is α-computable. Then $\mathrm{dom}(e)$ is r.e. and hence e is γ_e-computable with a tracking function

$$\bar{e}(i) \simeq \begin{cases} \ulcorner v_i \urcorner & \text{if } e(i)\downarrow \\ \uparrow & \text{otherwise.} \end{cases}$$

Furthermore $\gamma_e \leq \alpha$ by Theorem 2.19. The converse is trivial. □

3 Effective metric spaces and partial continuous functions

The typical metric algebra A is uncountable and we will approximate it with some countable metric subalgebra X which is effective. From X we will construct a set $\bar{X}_k^A$ consisting of those elements of A that can be computably approximated by elements of X. In this section we concentrate on effective metric spaces and partial functions on effective metric spaces. In the following sections we apply these results to the special case of effective metric partial algebras and partial homomorphisms between effective metric partial algebras.

3.1 Effective completion

Let $\mathbb{R}_k$ be the space of recursive reals and let $\rho: \Omega_\rho \to \mathbb{R}_k$ be a standard numbering of $\mathbb{R}_k$. This means that the metric on $\mathbb{R}$ is ρ-computable. We also consider some (recursively unique) computable numbering of the ordered field of rationals $\mathbb{Q}$ and a computable embedding into $\mathbb{R}_k$. We will, in the usual manner, not make this numbering and embedding explicit.

The following definition is given by Lacombe [17] and Moschovakis [20].

Definition 3.1. A metric space (X, d) is *effective* (or *recursive*) if there is a numbering $\alpha: \Omega_\alpha \to X$ such that

(i) $d: X \times X \to \mathbb{R}_k$; and

(ii) d is (α, ρ)-computable.

The effective metric space with numbering α is denoted by (X, α).

We now consider the case when (A, d) is a metric space, X is a subspace of A and (X, α) is effective. A sequence $x: \mathbb{N} \to X$ is an *α-computable Cauchy sequence* if x is α-computable and there is a recursive modulus function $m: \mathbb{N} \to \mathbb{N}$ such that for each $N \in \mathbb{N}$ and $i, j \in \mathbb{N}$,

$$i, j \geq m(N) \implies d_A(x(i), x(j)) < 2^{-N}.$$

The *computable completion* $\bar{X}^A_{k,\alpha}$ of (X, α) in A is defined by

$$\bar{X}^A_{k,\alpha} = \{x \in A : x \text{ limit of an } \alpha\text{-computable Cauchy sequence}\}.$$

An α-computable Cauchy sequence x with recursive modulus m is indexed by $\langle \bar{x}, \bar{m} \rangle$, where $\bar{x}$ is a recursive index for x and $\bar{m}$ is a recursive index for m. We let $\Omega^A_{\bar{\alpha}}$ be the set of indices for α-computable Cauchy sequences which converge to elements in A. Then we define $\bar{\alpha}: \Omega^A_{\bar{\alpha}} \to \bar{X}^A_{k,\alpha}$ by $\bar{\alpha}(\langle \bar{x}, \bar{m} \rangle) = \lim_{n\to\infty} x(n)$.

An α-computable sequence x is said to be a *fast α-computable Cauchy sequence* if $m \geq n \implies d(x(m), x(n)) < 2^{-n}$. Clearly, each fast α-computable Cauchy sequence is an α-computable Cauchy sequence with the identity function as modulus. Furthermore there is an algorithm which given an index for an α-computable Cauchy sequence x computes an index for an α-computable subsequence of x which is a fast Cauchy sequence. Thus α-computable Cauchy sequences and fast α-computable Cauchy sequences are computationally equivalent in a uniform way.

The following is well known and easy to prove.

Lemma 3.2. *Let (X, α) be an effective metric space and suppose X is a subspace of the metric space A. Then $(\bar{X}^A_{k,\alpha}, \bar{\alpha})$ is an effective metric space. Furthermore, the inclusion mapping $\iota: X \to \bar{X}^A_{k,\alpha}$ is $(\alpha, \bar{\alpha})$-computable.*

A limit algorithm takes a computable Cauchy sequence to its limit, if it exists. To make this precise, let (X, α) be an effective metric space and let $\text{Cauchy}_\alpha(X)$ be the set of α-computable Cauchy sequences. Let $\Omega_{c,\alpha}$ be the set of indices for α-computable Cauchy sequences and let $\alpha_c: \Omega_{c,\alpha} \to \text{Cauchy}_\alpha(X)$ be defined by

$$\alpha_c(\langle \bar{x}, \bar{m} \rangle) = \lambda n.\alpha(\{\bar{x}\}(n)).$$

Thus $(\text{Cauchy}_\alpha(X), \alpha_c)$ is a numbered set. Let $\lim: \text{Cauchy}_\alpha(X) \to X$ be the partial function defined by $\lim(x) \simeq \lim_{n\to\infty} x(n)$.

Definition 3.3. The effective metric space (X, α) has a *limit algorithm* if lim is weakly (α_c, α)-computable.

This is condition (A) in Moschovakis [20]. By the remarks before the previous lemma it is clear that it is equivalent to consider a limit algorithm working on fast Cauchy sequences.

Lemma 3.4. *Let (X, α) be an effective metric space and suppose X is a subspace of the metric space A. Then $(\bar{X}^A_{k,\alpha}, \bar{\alpha})$ has a limit algorithm.*

Proof. Let f be a partial recursive function such that if $\langle \bar{x}, \bar{m} \rangle \in \Omega_{c,\alpha}$ then $\lambda n.\alpha(\{f(\langle \bar{x}, \bar{m} \rangle)\}(n))$ is a fast α-computable subsequence of $\lambda n.\alpha(\{\bar{x}\}(n))$. Let g be a primitive recursive function such that

$$\{g(\langle \bar{x}, \bar{m} \rangle)\}(n) \simeq \{f(\{\bar{x}\}(n))\}(n),$$

and let h be a primitive recursive function such that

$$\{h(\bar{m})\}(N) \simeq \text{some } k[k > N + 2 \text{ and } k > \{\bar{m}\}(N + 2)].$$

Then we define $\overline{\lim}$ by

$$\overline{\lim}(\langle \bar{x}, \bar{m} \rangle) \simeq \langle g(\langle \bar{x}, \bar{m} \rangle), h(\bar{m}) \rangle.$$

Now suppose $\lim(\bar{\alpha}_c(\langle \bar{x}, \bar{m} \rangle))\!\downarrow = x \in A$. Then for each n, $\{\bar{x}\}(n) = \langle \bar{x}_n, \bar{m}_n \rangle$ is an $\bar{\alpha}$-index for $x_n \in \bar{X}^A_{k,\alpha}$. At the same time $\langle \bar{x}_n, \bar{m}_n \rangle$ is an α_c-index for an α-computable Cauchy sequence $(y_{nk})_k$ in X with computable modulus m_n. Thus

$$y_{nn} = \alpha(\{f(\langle \bar{x}_n, \bar{m}_n \rangle)\}(n)) = \alpha(\{g(\langle \bar{x}, \bar{m} \rangle)\}(n)).$$

Furthermore $\lambda N.\{\bar{m}\}(N)$ is a modulus for the sequence (x_n), and $\{h(\bar{m})\}$ is a total function. Note that for $n, m \geq \{h(\bar{m})\}(N)$,

$$d(y_{nn}, y_{mm}) \leq d(y_{nn}, x_n) + d(x_n, x_m) + d(x_m, y_{mm}) < 2^{-N}.$$

It follows that $(y_{nn})_n$ is an α-computable Cauchy sequence and its index is $\langle g(\langle \bar{x}, \bar{m} \rangle), h(\bar{m}) \rangle$ and (similarly) $\bar{\alpha}(\langle g(\langle \bar{x}, \bar{m} \rangle), h(\bar{m}) \rangle) = x$. □

Suppose X is a subset of a metric space (A, d). Then for $x \in A$ and $r \in \mathbb{N}$ we define

$$B^X(a, 2^{-r}) = \{x \in X : d(a, x) < 2^{-r}\},$$

the ball in X around a of radius 2^{-r}. If the superscript is deleted we mean the ball with respect to the whole space A.

Definition 3.5. Let (X, α) be an effective metric space. A subset $Y \subseteq X$ is α-*effectively dense* if there is a partial recursive function $g : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ such that given any $n \in \Omega_\alpha$ and $r \in \mathbb{N}$, then $g(n,r)\downarrow$, $g(n,r) \in \Omega_\alpha$ and

$$\alpha(g(n,r)) \in B^Y(\alpha(n), 2^{-r}).$$

Definition 3.6. An effective metric space (X, α) is α-*effectively separable* if it contains an α-computably enumerable set Y that is dense in X.

Note that a dense α-computably enumerable set is effectively dense.

Lemma 3.7. *Let X be a subspace of the metric space A and suppose (X, α) is an effective metric space. Then X is $\bar{\alpha}$-effectively dense in $\bar{X}^A_{k,\alpha}$.*

Proof. Define a partial recursive function g by

$$g(\langle \bar{x}, \bar{m} \rangle, r) \simeq \{\bar{x}\}(\{\bar{m}\}(\text{some } N(2^{-N} < 2^{-r}))),$$

where $r \in \mathbb{N}$. Clearly, $\alpha(g(\langle \bar{x}, \bar{m} \rangle, r)) \in B^X(\bar{\alpha}(\langle \bar{x}, \bar{m} \rangle), 2^{-r})$ when $\langle \bar{x}, \bar{m} \rangle \in \Omega^A_{\bar{\alpha}}$. Using the computable embedding of (X, α) into $(\bar{X}^A_{k,\alpha}, \bar{\alpha})$ we obtain the result. □

Reductions between effective numberings on completions are determined by reductions on effectively dense subspaces.

Lemma 3.8. *Let X be a metric space and let $Y \subseteq X$. Suppose X is numbered effectively as a metric space by α and β such that β has a limit algorithm, Y is α-effectively dense, and α reduces recursively to β on Y. Then α reduces recursively to β on X.*

Proof. Let f be a partial recursive function witnessing $\alpha \leq \beta$ on Y. Using the α-effective density of Y there is a partial recursive function $h(n,t)$ such that if $n \in \Omega_\alpha$ then

$$y_t = \alpha(h(n,t))$$

is an α-computable sequence in Y with modulus, say, $m(N) = 2^{-N}$. But then (y_t) is a β-computable Cauchy sequence tracked by $f(h(n,t))$. Note that a β_c-index for (y_t) is obtained uniformly in n. Now, using the limit algorithm we compute a β-index for $\alpha(n)$. Thus $\alpha \leq \beta$ on X. □

In particular, if α and β both have limit algorithms, Y is both α- and β-effectively dense, and α is recursively equivalent to β on Y, then α is recursively equivalent to β on X.

A useful result (Kushner [16]) is that, for important spaces, effective separability is inherited by semicomputable sets. We prove it here in our setting. The proof is an example of the "capture method", previously used by Ceitin. It is a simple but beautiful method combining effective separability with a limit algorithm. The substance of one version of the method is contained in the following lemma.

Lemma 3.9. *Let (X, α) be an effective metric space with a limit algorithm and let $x \in X$. Suppose $V \subseteq \Omega_\alpha$ is an r.e. set so that $x \in \overline{\alpha(V)}$, the closure of $\alpha(V)$. Then there is a total recursive function* cap, *obtained uniformly from x and V, such that*

$$\begin{cases} e \notin W_e & \Longrightarrow \quad \alpha(\text{cap}(e)) = x \\ e \in W_e & \Longrightarrow \quad \text{cap}(e) \in V. \end{cases}$$

Proof. Since $x \in \overline{\alpha(V)}$ we obtain, uniformly in x and V, an α-computable fast Cauchy sequence (x_n) converging to x, where $x_n \in \alpha(V)$ for each n. Define v by $v(e) \simeq$ least $k[e \in W_e^k]$ and let $f: \mathbb{N} \times \mathbb{N} \to X$ be the total α-computable function defined by

$$f(e,k) = \begin{cases} x_k & \text{if } e \notin W_e^k \\ x_{v(e)} & \text{if } e \in W_e^k. \end{cases}$$

Note that $\lambda k. f(e,k)$ is an α-computable fast Cauchy sequence for each e, with an index $s(e)$ obtained uniformly from e. Now define cap by $\text{cap}(e) = \overline{lim}(s(e))$. □

Theorem 3.10. *Let (X, α) be an effective metric space with a limit algorithm. If $S \subseteq X$ is α-semicomputable and $D \subseteq X$ is α-computably enumerable and dense in X then $D \cap S$ is α-computably enumerable and dense in S.*

In particular, if (X, α) has a limit algorithm and is effectively separable then every α-semicomputable set $S \subseteq X$ is α-effectively separable.

Proof. Let W be an r.e. set such that $\alpha^{-1}(S) = \Omega_\alpha \cap W$, and let $V \subseteq \Omega_\alpha$ be an r.e. set such that $\alpha(V)$ is dense in X. We will prove that $\alpha(W \cap V)$ is dense in S.

Let $x \in S$ and suppose we are given $k \in \mathbb{N}$. We need to show the existence of $n \in W \cap V$ such that $d_X(x, \alpha(n)) < 2^{-k}$. Let cap be the function from Lemma 3.9 for x and V and consider the set

$$U = \{e \in \mathbb{N} : d_X(\alpha(\text{cap}(e)), x) < 2^{-k} \wedge \text{cap}(e) \in W\}.$$

Let e be an r.e. index for U. If $e \notin U$ then $\alpha(\text{cap}(e)) = x$ so $e \in U$ by the definition of U. Thus $e \in U$, $\text{cap}(e) \in V \cap W$, and $d_X(\alpha(\text{cap}(e)), x) < 2^{-k}$. □

Observe that an α-index for the α-computably enumerable set $D \cap S$ is obtained uniformly from an α-index of the α-semicomputable set S and an α-index for the α-computably enumerable set D.

Corollary 3.11. *Let (X, α) and (Y, β) be effective metric spaces and suppose X is α-effectively separable and has a limit algorithm. If $h: X \to Y$ is a partial (α, β)-computable function then* $\text{dom}(h)$ *is α-effectively separable.*

The corollary follows since $\text{dom}(h)$ is α-semicomputable. Note that the corollary fails for weakly computable functions.

We close this section by considering some notions of sets being effectively open.

Definition 3.12. Let (X, α) be an effective metric space.

(i) A set $U \subseteq X$ is *α-effectively open* if there is a partial recursive function op such that for $n \in \Omega_\alpha$,

  (a) $\text{op}(n)\downarrow \iff \alpha(n) \in U$; and

  (b) $\text{op}(n)\downarrow \implies B(\alpha(n), 2^{-op(n)}) \subseteq U$.

  An index of op is an *index* for the α-effectively open set U.

(ii) A set $U \subseteq X$ is *α-Lacombe open* if there is an r.e. set W such that

$$U = \bigcup_{\langle n,r\rangle \in W} B^X(\alpha(n), 2^{-r}).$$

  An r.e. index of W is an *index* for the α-Lacombe open set U. (It is, implicitly, required that if $\langle n, r\rangle \in W$ then $n \in \Omega_\alpha$.)

Note that an index for an α-effectively open set U is also an index for U as an α-semicomputable set. It is straight forward to see that a Lacombe open set U is effectively open and that an index for U as an effectively open set is obtained uniformly from an index for U as a Lacombe open set. For the converse we need further assumptions.

Lemma 3.13. (Moschovakis [20]) *Let (X, α) be an effective metric space which is α-effectively separable and has a limit algorithm. Then every α-effectively open set U is α-Lacombe open. An index for U as an α-Lacombe open set is obtained uniformly from an index for U as an α-effectively open set.*

Proof. Let $U \subseteq X$ be α-effectively open and let op be the associated partial recursive function. Then U is α-semicomputable and hence α-effectively separable by Theorem 3.10. Let $V \subseteq \Omega_\alpha$ be an r.e. set such that $\alpha(V)$ is dense in U and let $W = \{\langle n, \text{op}(n)\rangle : n \in V\}$. We claim that

$$U = \bigcup_{\langle n,r\rangle \in W} B(\alpha(n), 2^{-r}).$$

To prove the nontrivial inclusion suppose $x \in U$ and let cap be the function from Lemma 3.9 for x and V. Let e be an index of the r.e. set

$$\{n \in \mathbb{N} : d_X(\alpha(\text{cap}(n)), x) < 2^{-\text{op}(\text{cap}(n))}\}.$$

If $e \notin W_e$ then $\alpha(\text{cap}(e)) = x$ so $e \in W_e$. Thus $e \in W_e$. But then $\text{cap}(e) \in V$, $\langle \text{cap}(e), \text{op}(\text{cap}(e)) \rangle \in W$, and

$$x \in B(\alpha(\text{cap}(e)), 2^{-\text{op}(\text{cap}(e))}).$$

The claimed uniformity follows from the uniformity of Theorem 3.10. □

3.2 Ceitin's theorem

Ceitin [5] proved that a computable function between effective metric spaces, which satisfy certain conditions, is effectively continuous. In this section we use his theorem in the version given in Moschovakis [20] to study the relationship between computability and continuity in our setting. We review this result and related results from Moschovakis [20]. In proving these basic theorems we adapt the constructive methods in Kushner [16] to our setting.

First we define several notions of effective continuity.

Definition 3.14. Let (X, α) and (Y, β) be effective metric spaces with metrics d_X and d_Y, respectively. Let $h: X \to Y$ be a partial function.

(i) h is *α-effectively continuous* if there is a partial recursive function $r(n, N)$ such that if $x = \alpha(n) \in \text{dom}(h)$ then $r(n, N)\downarrow$ and for each $y \in \text{dom}(h)$,

$$d_X(x, y) < 2^{-r(n,N)} \implies d_Y(h(x), h(y)) < 2^{-N}.$$

(ii) h is *(α, β)-Lacombe continuous* if there is a partial recursive function $f(m, n)$ such that if $m \in \Omega_\beta$ then $f(m, N)\downarrow$ and

$$h^{-1}(B(\beta(m), 2^{-N})) = \bigcup_{\langle n,r \rangle \in W_{f(m,N)}} B^{\text{dom}(h)}(\alpha(n), 2^{-r}).$$

It is implicitly assumed that if $m \in \Omega_\beta$ and $\langle n, r \rangle \in W_{f(m,N)}$ then $n \in \Omega_\alpha$.

(iii) h is *effectively uniformly continuous* if there is a total recursive function $r(N)$ such that for all $x, y \in \text{dom}(h)$,

$$d_X(x, y) < 2^{-r(N)} \implies d_Y(h(x), h(y)) < 2^{-N}.$$

Obviously every effectively uniformly continuous function is effectively continuous.

Lemma 3.15. *Let (X, α) and (Y, β) be effective metric spaces and assume (Y, β) is β-effectively separable. If the partial function $h: X \to Y$ is (α, β)-Lacombe continuous then h is α-effectively continuous.*

Proof. Let f be the partial recursive function witnessing that h is (α,β)-Lacombe continuous and let $V \subseteq \Omega_\beta$ be an r.e. set such that $\beta(V)$ is dense in Y. Define $r(n,N)$ by

$$r(n,N) \simeq \text{some } t[(\exists m \in V)(\exists \langle k,s\rangle \in W_{f(m,N+1)}) \\ (d_X(\alpha(n),\alpha(k)) + 2^{-t} < 2^{-s})].$$

Then $r(n,N)$ is a partial recursive function by using, as usual, the partial recursive tracking functions for d_X and $+$, and the r.e. tracking relation for $<$. Suppose $x = \alpha(n) \in \mathrm{dom}(h)$ and let $m \in V$ be such that $d_Y(h(x),\beta(m)) < 2^{-N-1}$. Then $x \in B(\alpha(k),2^{-s})$ for some $\langle k,s\rangle \in W_{f(m,N+1)}$ so $r(n,N)\downarrow$. Suppose further that $y \in \mathrm{dom}(h)$ and $d_X(\alpha(n),y) < 2^{-r(k,N)}$. Let m and $\langle k,s\rangle$ be witnesses to $r(n,N) \simeq t$. Then

$$y \in B^{\mathrm{dom}(h)}(x,2^{-t}) \subseteq B^{\mathrm{dom}(h)}(\alpha(k),2^{-s})$$

and hence $h(x),h(y) \in B^Y(\beta(m),2^{-N-1})$, i.e. $d_Y(h(x),h(y)) < 2^{-N}$. □

By redefining r in the above proof slightly one can replace effective separability of (Y,β) by h being weakly (α,β)-computable.

Lemma 3.16. *Let (X,α) and (Y,β) be effective metric spaces and assume the partial function $h\colon X \to Y$ is weakly (α,β)-computable. If h is (α,β)-Lacombe continuous then h is α-effectively continuous.*

Our goal now is to prove Ceitin's theorem: a computable partial function from X to Y is Lacombe continuous when the space X is effectively separable and has a limit algorithm. In proving Ceitin's theorem we use the following important separation theorem due to Moschovakis [20]. Moschovakis used the second recursion theorem in his proof. Here we follow Kushner [16] and use the capture method, although in a slightly more subtle form than Lemma 3.9.

Definition 3.17. Let (X,α) be an effective metric space. A set $T \subseteq X$ is said to be *trackable* if there is a partial recursive function tr such that if $n \in \Omega_\alpha$ and $B(\alpha(n),2^{-k}) \cap T \neq \emptyset$ then $\mathrm{tr}(n,k)\downarrow$ and $\alpha(\mathrm{tr}(n,k)) \in B(\alpha(n),2^{-k}) \cap T$. The function tr is a *tracking function* for T.

It is easy to see that every α-computably enumerable subset of an effective metric space (X,α) is trackable uniformly in the set, i.e. an index for a tracking function is obtained uniformly from an index for an α-computable enumeration. Furthermore, if $S \subseteq X$ is α-semicomputable and contains an α-computably enumerable subset $D \subseteq S$ dense in S, then S is trackable uniformly in (indices for) D and S.

Lemma 3.18. *Let (X, α) be an effective metric space with a limit algorithm and suppose (X, α) is α-effectively separable. Then every α-semicomputable set $S \subseteq X$ is trackable with a tracking function obtained uniformly from an α-index for S. More precisely, there is a partial recursive function* $\mathrm{tr}(e, n, N)$ *such that if e is an α-index for S then $\lambda nk.\mathrm{tr}(e, n, N)$ is a tracking function for S.*

Proof. By Theorem 3.10 there is an α-computably enumerable set $D \subseteq S$ dense in S, obtained uniformly from S. □

Here is the Moschovakis separation theorem.

Theorem 3.19. *Let (X, α) be an effective metric space with a limit algorithm. There is a partial recursive function* $\mathrm{sep}(a, t, n)$ *such that if $T \subseteq X$ is trackable with a tracking function with index t, $S \subseteq X$ is α-semicomputable with index a, and $S \cap T = \emptyset$, then for each $n \in \Omega_\alpha$ such that $\alpha(n) \in S$,* $\mathrm{sep}(a, t, n)\downarrow$ *and*

$$B(\alpha(n), 2^{-\mathrm{sep}(a,t,n)}) \cap T = \emptyset.$$

Proof. Let $v(e) \simeq$ least $i[e \in W_e^i]$ and define

$$f(e, t, n, k) \simeq \begin{cases} n & \text{if } e \notin W_e^k \\ \{t\}(n, v(e)) & \text{if } e \in W_e^k. \end{cases}$$

Let s be a primitive recursive function such that $\{s(e, t, n)\}(k) \simeq f(e, t, n, k)$ and define g by $g(e, t, n) \simeq \overline{\lim}(s(e, t, n))$ where $\overline{\lim}$ is a tracking function for the limit algorithm for fast Cauchy sequences. Let u be a primitive recursive function such that

$$W_{u(a,t,n)} = \{e : g(e, t, n)\downarrow \in W_a\}$$

and define sep by

$$\mathrm{sep}(a, t, n) \simeq v(u(a, t, n)).$$

Now suppose $S \subseteq X$ is α-semicomputable with index a, $T \subseteq X$ has a tracking function with index t, $S \cap T = \emptyset$, and suppose $x = \alpha(n) \in S$. If $u(a, t, n) \notin W_{u(a,t,n)}$ then $\lambda k.f(u(a, t, n), t, n, k)$ is total recursive and $\alpha g(u(a, t, n), t, n) = x \in S$. But then $u(a, t, n) \in W_{u(a,t,n)}$. It follows that $u(a, t, n) \in W_{u(a,t,n)}$ and $\mathrm{sep}(a, t, n)\downarrow$.

Suppose, to obtain a contradiction, that $B(x, 2^{-\mathrm{sep}(a,t,n)}) \cap T \neq \emptyset$. Then

$$y = \alpha(\{t\}(n, v(u(a, t, n))))\downarrow \in B(x, 2^{-\mathrm{sep}(a,t,n)}) \cap T.$$

But then $\lambda k.\alpha f(u(a, t, n), t, n, k)$ is an α-computable fast Cauchy sequence converging to y, i.e. $\alpha g(u(a, t, n), t, n) = y$. But $S \cap T = \emptyset$ so $y \notin S$ and hence we conclude that $g(u(a, t, n), t, n) \notin W_a$ and hence $u(a, t, n)\downarrow \notin W_{u(a,t,n)}$. □

Here is a first approximation to Ceitin's theorem.

Theorem 3.20. *Let (X,α) and (Y,β) be effective metric spaces and suppose X is α-effectively separable and has a limit algorithm. Then every partial (α,β)-computable function $h: X \to Y$ is α-effectively continuous.*

Proof. Assume that the partial function $h: X \to Y$ is (α,β)-computable. Then, in particular, $\mathrm{dom}(h)$ is α-semicomputable. Thus, for $x \in \mathrm{dom}(h)$ and $N \in \mathbb{N}$, the sets

$$A_{x,N} = \{y \in \mathrm{dom}(h) : d_Y(h(x), h(y)) < 2^{-N-1}\}$$

and

$$C_{x,N} = \{y \in \mathrm{dom}(h) : d_Y(h(x), h(y)) > 2^{-N-1}\}$$

are α-semicomputable, uniformly in x and N.

Let $s_1(n,N)$ and $s_2(n,N)$ be primitive recursive functions giving α-semicomputable indices for $A_{x,N}$ and $C_{x,N}$, respectively, when $x = \alpha(n) \in \mathrm{dom}(h)$. Let tr be the partial recursive function from Lemma 3.18 for (X,α) and let $\bar{s}$ be a primitive recursive function such that $\{\bar{s}(n,N)\}(m,k) \simeq \mathrm{tr}(s_2(n,N), m, k)$. Let sep be the partial recursive function from Theorem 3.19 and define r by

$$r(n,N) \simeq \mathrm{sep}(s_1(n,N), \bar{s}(n,N), n).$$

Then, for $x = \alpha(n) \in \mathrm{dom}(h)$, $\bar{s}(n,N)$ is an index for a tracking function for $C_{x,N}$. $A_{x,N}$ and $C_{x,N}$ are disjoint so $r(n,N)\!\downarrow$ and $B(x, 2^{-r(n,N)}) \cap C_{x,N} = \emptyset$. It follows that if $y \in \mathrm{dom}(h)$ and $d_X(x,y) < 2^{-r(n,N)}$ then $d_Y(h(x), h(y)) \leq 2^{-N-1} < 2^{-N}$. □

Theorem 3.21. (Ceitin) *Let (X,α) and (Y,β) be effective metric spaces and suppose X is α-effectively separable and has a limit algorithm. Then every partial (α,β)-computable function $h: X \to Y$ is (α,β)-Lacombe continuous.*

Proof. We may without loss of generality assume that h is a total function since $\mathrm{dom}(h)$ is α-semicomputable and hence, by Theorem 3.10, α-effectively separable. Furthermore the limit algorithm is inherited by any subspace.

We will show that $h^{-1}(B(\beta(m), 2^{-N}))$ is α-effectively open uniformly in m and N, for $m \in \Omega_\beta$. Lemma 3.13 then implies that $h^{-1}(B(\beta(m), 2^{-N}))$ is α-Lacombe open uniformly in m and N, for $m \in \Omega_\beta$, and hence that h is (α,β)-Lacombe continuous.

Let r be the partial recursive function witnessing the α-effective continuity of h (by Theorem 3.20). The set

$$A_{y,N} = \{x \in \mathrm{dom}(h) : d_Y(y, h(x)) < 2^{-N}\}$$

is α-semicomputable, uniformly in $y \in Y$ and $N \in \mathbb{N}$. Let $s(m,N)$ be a primitive recursive function giving an α-semicomputable index for $A_{y,N}$

when $y \simeq \beta(m)$. There is a $(\beta \times \mathrm{id} \times \alpha, \mathrm{id})$-computable function $g: Y \times \mathbb{N} \times X \to \mathbb{N}$ satisfying

$$g(y, N, x) \simeq \text{ some } t[d_Y(y, h(x)) + 2^{-t} < 2^{-N}].$$

Let $\bar{g}(m, N, n)$ be a partial recursive tracking function for g and define $\mathrm{op}(m, N, n)$ by

$$\mathrm{op}(m, N, n) \simeq r(n, \bar{g}(m, N, n)) \text{ if } n \in W_{s(m,N)}.$$

Fix $m \in \Omega_\beta$ and $N \in \mathbb{N}$ and suppose $n \in \Omega_\alpha$. If $\mathrm{op}(m, N, n)\downarrow$ then $n \in W_{s(m,N)}$ and $\alpha(n) \in h^{-1}(B(\beta(m), 2^{-N}))$. The converse is similar. Now suppose $\mathrm{op}(m, N, n)\downarrow$ and $y \in B(\alpha(n), 2^{-\mathrm{op}(m,N,n)})$, i.e. $d_X(\alpha(n), y) < 2^{-r(n, g(m,N,n))}$. Then

$$d_Y(h(\alpha(n)), h(y)) < 2^{-g(m,N,n)}$$

and hence, by the triangle inequality,

$$d_Y(\beta(m), h(y)) < 2^{-N}.$$

□

For a converse we need assumptions on the space (Y, β) and on the domain of h. First we observe the following.

Proposition 3.22. *Suppose (X, α) and (Y, β) are effective metric spaces and assume (Y, β) is β-effectively separable and has a limit algorithm. Let $h: X \to Y$ be a partial function. If h is (α, β)-Lacombe continuous and* $\mathrm{dom}(h)$ *is α-semicomputable then h is (α, β)-computable.*

Proof. Let W be an r.e. set witnessing that $\mathrm{dom}(h)$ is α-semicomputable and let $V \subseteq \Omega_\beta$ be an r.e. set such that $\beta(V)$ is dense in Y. Let f be the partial recursive function witnessing the (α, β)-Lacombe continuity of h. Let $\bar{s}$ be a partial recursive tracking function for the α-computable function $s: \mathrm{dom}(h) \times \mathbb{N} \to \mathbb{N}$ defined by

$$s(x, N) \simeq \text{some } m \in V[(\exists \langle k, r\rangle \in W_{f(m,N)})(d_X(x, \alpha(k)) < 2^{-r}].$$

Then define $\bar{h}$ by

$$\bar{h}(n) \simeq \begin{cases} \overline{\lim}(\lambda N.\bar{s}(n, N)) & \text{if } n \in W \\ \uparrow & \text{if } n \notin W. \end{cases}$$

We show that $\bar{h}$ tracks h. Suppose $n \in \Omega_\alpha$. If $\bar{h}(n)\downarrow$ then $\alpha(n) \in \mathrm{dom}(h)$, by definition. For the converse assume $\alpha(n) \in \mathrm{dom}(h)$ and fix N. By β-effective density there is $m \in V$ such that $d_Y(\beta(m), h(\alpha(n)) < 2^{-N}$ and hence, by (α, β)-Lacombe continuity, there is $\langle k, r\rangle \in W_{f(n,N)}$ such that $d_X(\alpha(k), \alpha(n)) < r$. Then, for $y_N = \beta(\bar{s}(n, N))$, we have $d_Y(y_N, h(\alpha(n))) < 2^{-N}$. Thus (y_N) is a β-computable fast Cauchy sequence approaching $h(\alpha(n))$, so $\bar{h}(n)\downarrow \in \Omega_\beta$ and $h(\alpha(n)) \simeq \beta(\bar{h}(n))$. □

Collecting both directions we obtain

Theorem 3.23. *Let (X, α) and (Y, β) be effective metric spaces which are effectively separable and have limit algorithms. Let $h: X \to Y$ be a partial function. Then h is (α, β)-computable if, and only if, h is (α, β)-Lacombe continuous and* $\mathrm{dom}(h)$ *is α-semicomputable.*

4 Continuity and computability of partial homomorphisms

Now we combine algebra, topology and effectivity to prove the main theorem stated in the introduction and to indicate some of its applications.

4.1 Effective metric partial Σ-algebras

In this section we make some initial remarks on the effective content of combining the metric with the algebraic structure.

Definition 4.1. A *metric partial Σ-algebra* is a partial Σ-algebra A equipped with a metric $d: A \times A \to \mathbb{R}$ such that each partial operation of A is continuous in the topology defined by the metric.

We have the following easy lifting result. Its effective analogue, which we do not consider in this paper, is much more subtle.

Lemma 4.2. *Let A be a metric partial Σ-algebra such that* $\mathrm{dom}(\sigma)$ *is open for each operation σ of A. Let X be a metric partial Σ-subalgebra of A. Then the closure $\bar{X}^A$ of X in A is a metric partial Σ-subalgebra of A containing X.*

We add effectivity in the obvious way.

Definition 4.3. A metric partial Σ-algebra X is (*weakly*) *effective* with a numbering α if

(i) (X, α) is a (weakly) effective partial Σ-algebra;

(ii) (X, α) is an effective metric space.

The following proposition shows that if e is an α-computable generating sequence of (X, α) then γ_e suffices for the effective completion, i.e., $\bar{\gamma}_e \sim \bar{\alpha}$.

Proposition 4.4. *Let A be a metric partial Σ-algebra and let X be a metric partial Σ-subalgebra of A. Let α be a numbering of X such that (X, α) is an effective metric partial Σ-algebra. Suppose $e: \mathbb{N} \to X$ is a partial α-computable sequence such that $\langle e \rangle$ is dense in X. Then the following hold:*

(i) *$(\langle e \rangle, \gamma_e)$ is an effective metric partial Σ-subalgebra of X and Ω_e is r.e.*

(ii) *The inclusion* $\iota\colon \langle e\rangle \to X$ *is* (γ_e, α)*-computable.*

(iii) $(\bar{X}^A_{k,\alpha}, \bar{\alpha})$ *is effectively separable.*

(iv) $\overline{\langle e\rangle}^A_{k,\gamma_e} = \bar{X}^A_{k,\alpha}$.

(v) $\bar{\gamma}_e \sim \bar{\alpha}$.

Proof. (i) and (ii) follows from Theorem 2.19, noting that $\gamma_e \leq \alpha$ so that the metric on $\langle e\rangle$ is γ_e-computable. Let $\bar{\iota}_1$ and $\bar{\iota}_2$ be tracking functions showing that the inclusions $\iota_1\colon \langle e\rangle \to X$ and $\iota_2\colon X \to \bar{X}^A_{k,\alpha}$ are (γ_e, α)-computable and $(\alpha, \bar{\alpha})$-computable, respectively. Then $\bar{\iota}_2\bar{\iota}_1(\Omega_e)$ is an r.e. set witnessing that $\bar{X}^A_{k,\alpha}$ is $\bar{\alpha}$-effectively separable.

We prove the non-trivial inclusion in (iv) using an informal description of an algorithm. It is routinely formalised to also prove (v). Given $x \in \bar{X}^A_{k,\alpha}$ we effectively obtain (an index for) an α-computable fast Cauchy sequence (x_n) converging to x. For each n find effectively some $k_n \in \Omega_e$ such that

$$d_X(\gamma_e(k_n), x_n) < 2^{-n}.$$

Thus we obtain a γ_e-computable sequence (y_n) in $\langle e\rangle$, where $y_n = \gamma_e(k_n)$. By the triangle inequality,

$$d_A(x, y_n) \leq d_A(x, x_n) + d_A(x_n, y_n) < 2^{-(n-1)}$$

proving that $y_n \to x$ with a computable modulus $m(N) = N - 1$. This shows $x \in \overline{\langle e\rangle}^A_{k,\gamma_e}$. □

4.2 Equivalence theorem

In this section we prove the main result connecting computability and continuity for partial homomorphisms between metric partial Σ-algebras. We assume throughout that A and B are metric partial Σ-algebras and that X and Y are metric partial Σ-subalgebras of A and B, respectively.

Our first result only deals with effective metric spaces.

Lemma 4.5. *Suppose* (X, α) *and* (Y, β) *are effective metric spaces and* (X, α) *is effectively separable. Let* $h\colon \bar{X}^A_{k,\alpha} \to B$ *be a partial function such that* $h(X) \subseteq Y$ *and* $\mathrm{dom}(h)$ *is* $\bar{\alpha}$*-semicomputable. If* h *is* $\bar{\alpha}$*-effectively continuous and* $h|_X\colon X \to Y$ *is weakly* (α, β)*-computable then* h *is* $(\bar{\alpha}, \bar{\beta})$*-computable. In particular,* $h(\bar{X}^A_{k,\alpha}) \subseteq \bar{Y}^B_{k,\beta}$.

Proof. Let V be an r.e. set such that $\alpha(V)$ is dense in X and let W be an r.e. set such that $\bar{\alpha}^{-1}(\mathrm{dom}(h)) = W \cap \Omega_{\bar{\alpha}}$.

We first show that $\mathrm{dom}(h)$ is α-effectively separable. Let $\iota_X\colon X \to \bar{X}^A_{k,\alpha}$ be the $(\alpha, \bar{\alpha})$-computable embedding of Lemma 3.2 with partial recursive tracking function $\bar{\iota}_X$. Define an r.e. set Z by

$$Z = \{n \in V : \bar{\iota}_X(n) \in W\}.$$

We claim that $\alpha(Z)$ is a dense subset of $\mathrm{dom}(h)$. The set $\bar{V} = \bar{\iota}_X(V) \subseteq \Omega_{\bar{\alpha}}$ is r.e. and $\bar{\alpha}(\bar{V})$ is dense in $\bar{X}^A_{k,\alpha}$. By Theorem 3.10, $\bar{\alpha}(\bar{V} \cap W)$ is dense in $\mathrm{dom}(h)$. But $\bar{\iota}_X(Z) = \bar{V} \cap W$ and $\alpha(n) = \bar{\alpha}(\bar{\iota}_X(n))$ for each $n \in Z$. Thus $\alpha(Z)$ is dense in $\mathrm{dom}(h)$.

Let the partial recursive function r witness the $\bar{\alpha}$-effective continuity of h. We may assume $r(\langle \bar{x}, \bar{m}\rangle, N)$ is strictly increasing in N, when $\bar{\alpha}(\langle \bar{x}, \bar{m}\rangle) \in \mathrm{dom}(h)$. Define a partial recursive function s which for $\langle \bar{x}, \bar{m}\rangle \in \Omega^A_{\bar{\alpha}}$ satisfies

$$s(\langle \bar{x}, \bar{m}\rangle, N) \simeq (\text{some } n \in Z)[d_{\bar{X}^A_{k,\alpha}}(\bar{\alpha}(\langle \bar{x}, \bar{m}\rangle), \iota_X(\alpha(\{\bar{x}\}(n)))) < 2^{-r(\langle \bar{x}, \bar{m}\rangle, N+1)}].$$

Let t be a primitive recursive function such that

$$\{t(\langle \bar{x}, \bar{m}\rangle)\}(N) \simeq \bar{\iota}_Y(\tilde{h}(\{\bar{x}\}(s(\langle \bar{x}, \bar{m}\rangle, N)))),$$

where $\tilde{h}$ is a partial recursive tracking function for $h|_X$ and $\bar{\iota}_Y$ is a partial recursive tracking function for ι_Y. Then we define the partial recursive function $\bar{h}$ by

$$\bar{h}(\langle \bar{x}, \bar{m}\rangle) \simeq \overline{\lim}(t(\langle \bar{x}, \bar{m}\rangle)),$$

where $\overline{\lim}$ is a partial recursive function witnessing the limit algorithm for $\bar{Y}^B_{k,\beta}$ with respect to fast $\bar{\beta}$-computable sequences (by Lemma 3.4).

To show that $\bar{h}$ tracks h assume $\langle \bar{x}, \bar{m}\rangle \in \Omega^A_{\bar{\alpha}}$, $\bar{\alpha}(\langle \bar{x}, \bar{m}\rangle) = x$, and $x \in \mathrm{dom}(h)$. Then $r(\langle \bar{x}, \bar{m}\rangle, N)\downarrow$ and $\alpha(\{\bar{x}\}(s(\langle \bar{x}, \bar{m}\rangle, N)))\downarrow$ for each $N \in \mathbb{N}$. Let $x_N = \alpha(\{\bar{x}\}(s(\langle \bar{x}, \bar{m}\rangle, N)))$. Then $d_A(x, x_N) < 2^{-r(\langle \bar{x}, \bar{m}\rangle, N+1)}$ and hence $d_B(h(x), h(x_N)) < 2^{-N-1}$. It follows that $(h(x_N))$ is a fast $\bar{\beta}$-computable sequence converging to $h(x)$ by continuity. Thus

$$h(\bar{\alpha}(\langle \bar{x}, \bar{m}\rangle)) \simeq \bar{\beta}(\bar{h}(\langle \bar{x}, \bar{m}\rangle)).$$

□

We now state and prove our main theorem.

Theorem 4.6. *Let A and B be metric partial Σ-algebras and assume that (X, α) and (Y, β) are effective metric partial Σ-algebras such that X and Y are metric partial Σ-subalgebras of A and B, respectively. Let $e\colon \mathbb{N} \to X$ be a partial α-computable sequence such that $\langle e\rangle$ is dense in X.*

Suppose $h\colon \bar{X}^A_{k,\alpha} \to \bar{Y}^B_{k,\beta}$ is a partial function such that $he(\mathbb{N}) \subseteq Y$, $he\colon \mathbb{N} \to Y$ is a partial β-computable sequence, and $h|_{\langle e\rangle}\colon \langle e\rangle \to Y$ is a strong partial Σ-homomorphism. Then the following are equivalent:

(i) *h is $(\bar{\alpha}, \bar{\beta})$-computable;*

(ii) *h is $(\bar{\alpha}, \bar{\beta})$-Lacombe continuous and* $\mathrm{dom}(h)$ *is $\bar{\alpha}$-semicomputable;*

(iii) *h is $\bar{\alpha}$-effectively continuous and* $\mathrm{dom}(h)$ *is $\bar{\alpha}$-semicomputable.*

Proof. Note that $(\bar{X}^A_{k,\alpha}, \bar{\alpha})$ has a limit algorithm by Lemma 3.4 and is effectively separable by Proposition 4.4. Furthermore $h(\langle e\rangle) = \langle he\rangle$ by Lemma 2.6.

(i) $\implies$ (ii): If h is $(\bar{\alpha}, \bar{\beta})$-computable then $\mathrm{dom}(h)$ is $\bar{\alpha}$-semicomputable and, by Ceitin's Theorem 3.21, $(\bar{\alpha}, \bar{\beta})$-Lacombe continuous.

(ii) $\implies$ (iii): Assume (ii). Let $\tilde{\beta}$ be the restriction of $\bar{\beta}$ to $\bar{\beta}^{-1}(\mathrm{im}(h))$. Then h is also $(\bar{\alpha}, \tilde{\beta})$-Lacombe continuous. Hence, by Lemma 3.15, it suffices to show that $(\mathrm{im}(h), \tilde{\beta})$ is effectively separable. The set Ω_{he} is r.e. since he is a partial β-computable sequence. Using the $(\beta, \tilde{\beta})$-computable embedding $\iota_{\langle he\rangle} \colon \langle he\rangle \to \mathrm{im}(h)$ it therefore suffices to show that $\langle he\rangle$ is dense in $\mathrm{im}(h)$.

Let $m \in \Omega_{\tilde{\beta}}$ and $N \in \mathbb{N}$ and suppose $B(\tilde{\beta}(m), 2^{-N}) \cap \mathrm{im}(h) \neq \emptyset$. Let f be the partial recursive function witnessing that h is $(\bar{\alpha}, \tilde{\beta})$-Lacombe continuous. Then

$$h^{-1}(B(\tilde{\beta}(m), 2^{-N})) = \bigcup_{\langle n,r\rangle \in W_{f(m,N)}} B^{\mathrm{dom}(h)}(\alpha(n), 2^{-r}).$$

So there is $\langle n, r\rangle \in W_{f(m,N)}$ such that $B^{\mathrm{dom}(h)}(\alpha(n), 2^{-r}) \neq \emptyset$. By assumption, $\mathrm{dom}(h)$ is $\bar{\alpha}$-semicomputable. Furthermore $\bar{X}^A_{k,\alpha}$ is $\bar{\alpha}$-effectively separable under the dense set $\langle e\rangle$. It follows by Theorem 3.10 that $\langle e\rangle \cap \mathrm{dom}(h)$ is dense in $\mathrm{dom}(h)$. Thus there is an $x \in \langle e\rangle \cap \mathrm{dom}(h) \cap B(\alpha(n), 2^{-r})$. For such x,

$$h(x)\downarrow \in h(\langle e\rangle) \cap B(\tilde{\beta}(m), 2^{-N}) = \langle he\rangle \cap B(\tilde{\beta}(m), 2^{-N}),$$

that is $\langle he\rangle$ is dense in $\mathrm{im}(h)$.

(iii) $\implies$ (i): It suffices to consider $(\langle e\rangle, \gamma_e)$ by Proposition 4.4. We therefore apply Lemma 4.5 with $(\langle e\rangle, \gamma_e)$ in place of (X, α). Lemma 2.18 says that $h|_{\langle e\rangle} \colon \langle e\rangle \to \langle he\rangle$ is weakly (γ_e, γ_{he})-computable and hence (γ_e, β)-computable since $\gamma_{he} \leq \beta$. Thus all hypotheses in Lemma 4.5 are satisfied so h is $(\bar{\gamma}_e, \bar{\beta})$-computable. Then h is $(\bar{\alpha}, \bar{\beta})$-computable by Proposition 4.4. $\square$

Remark 4.7. (i) Note that in the case B is a total Σ-algebra and $h \colon A \to B$ is a partial Σ-homomorphism such that $\mathrm{im}(e) \subseteq \mathrm{dom}(h)$, then $\langle e\rangle \subseteq \mathrm{dom}(h)$ and $h|_{\langle e\rangle}$ is a strong homomorphism.

(ii) In mathematical applications (e.g. topological groups or Banach spaces) it is often the case that homomorphisms from A to B are continuous if, and only if, they are uniformly continuous. Recall that, in the notation above, if h is effectively uniformly continuous on A then h is $\bar{\alpha}$-effectively continuous on $\bar{X}^A_{k,\alpha}$ and hence (when h is a strong partial homomorphism on $\langle e\rangle$ and $\mathrm{dom}(h)$ is $\bar{\alpha}$-semicomputable) h is $(\bar{\alpha}, \bar{\beta})$-computable.

(iii) Suppose $h: A \to B$ is a partial Σ-homomorphism which is strong on $\langle e \rangle$ and such that he is a partial β-computable sequence in Y. If $h|_{\bar{X}^A_{k,\alpha}}$ is $\bar{\alpha}$-effectively continuous and $\mathrm{dom}(h|_{\bar{X}^A_{k,\alpha}})$ is $\bar{\alpha}$-semidecidable then $h(\bar{X}^A_{k,\alpha}) \subseteq \bar{Y}^B_{k,\beta}$, i.e., h takes computable elements to computable elements.

From the proof of the theorem and Proposition 4.4 we see that the role of α was only to ensure that Ω_e is r.e. and that the metric on $\langle e \rangle$ is γ_e-computable. Recall that, by Proposition 2.17, if Ω_e is r.e. then e is γ_e-computable and $(\langle e \rangle, \gamma_e)$ is an effective partial Σ-algebra.

Here is a reformulation of the main theorem in terms of γ_e.

Theorem 4.8. *Let A and B be metric partial Σ-algebras and let $e: \mathbb{N} \to A$ be a partial sequence such that Ω_e is r.e. and the metric on $\langle e \rangle$ is (γ_e, ρ)-computable.*

Suppose $h: A \to B$ is a partial function such that $h|_{\langle e \rangle}: \langle e \rangle \to B$ is a strong partial homomorphism, Ω_{he} is r.e., and the metric on $\langle he \rangle$ is (γ_{he}, ρ)-computable. Then the following are equivalent:

(i) *h is $(\bar{\gamma}_e, \bar{\gamma}_{he})$-computable;*

(ii) *$h|_{\overline{\langle e \rangle}^A_{k,\gamma_e}}$ is $\bar{\gamma}_e$-effectively continuous and its domain* $\mathrm{dom}(h|_{\overline{\langle e \rangle}^A_{k,\gamma_e}})$ *is $\bar{\gamma}_e$-semicomputable.*

4.3 Applications

In this section we briefly consider applications of our main theorem. First we apply it to Banach spaces and illuminate its relationship to the First Main Theorem of Pour-El and Richards [23]. Then we consider certain topological groups.

4.3.1 Banach spaces

Consider the main theorem in case A and B are real Banach spaces. (Complex Banach spaces are treated similarly.) A Banach space is, strictly speaking, a many sorted algebra, containing a space X and the field of real numbers $\mathbb{R}$. Given the single sorted version of the general theory we will give a single sorted treatment of Banach spaces as follows.

For real Banach spaces we consider the signature

$$\Sigma = \{+, 0, r : r \in \mathbb{Q}\},$$

where $+$ is addition on the space, 0 is the additive identity, and r is the unary operation for scalar multiplication, operating on elements in the space. We restrict the scalars in the signature Σ to $\mathbb{Q}$ in order to obtain an effective theory. Note that, due to the restriction to $\mathbb{Q}$, we have a uniform enumeration of $T(\Sigma, V)$.

A partial Σ-homomorphism $T: A \to B$ is a partial operator linear with respect to addition and the scalars in $\mathbb{Q}$, i.e., a $\mathbb{Q}$-linear map. In particular, partial $\mathbb{R}$-linear operators are partial Σ-homomorphisms.

Recall that a partial linear operator $T: A \to B$ is continuous if, and only if, T is bounded, i.e., there is a constant M such that for each $x \in \mathrm{dom}(T)$

$$||T(x)|| \leq M||x||.$$

In this case

$$||T(x) - T(y)|| = ||T(x - y)|| \leq M||x - y||.$$

It follows that T is effectively uniformly continuous on its domain when bounded.

We use the convention that a partial linear operator T is bounded on A if it is bounded on $\mathrm{dom}(T)$.

We will now state our version of the first main theorem in Pour-El and Richards [23].

Theorem 4.9. *Let A and B be Banach spaces with the signature Σ. Let (X, α) and (Y, β) be effective Σ-subalgebras of A and B respectively such that their norms are (α, ρ)-computable and (β, ρ)-computable, respectively. Suppose $e: \mathbb{N} \to X$ is a partial α-computable sequence such that $\langle e \rangle$ is a dense Σ-subalgebra of A.*

Let $T: A \to B$ be a partial linear operator such that $\mathrm{im}(e) \subseteq \mathrm{dom}(T)$ and $Te: \mathbb{N} \to Y$ is a partial β-computable sequence. Then the following are equivalent.

(i) *T is $(\bar{\alpha}, \bar{\beta})$-computable on $\bar{X}^A_{k,\alpha}$; and*

(ii) *T is bounded on $\bar{X}^A_{k,\alpha}$ and $\mathrm{dom}(T) \cap \bar{X}^A_{k,\alpha}$ is $\bar{\alpha}$-semicomputable.*

Proof. (X, α) and (Y, β) are effective metric spaces since their norms are computable. By Remark 4.7 (i), $T|_{\langle e \rangle}$ is a strong Σ-homomorphism. Furthermore T is bounded if, and only if, T is effectively uniformly continuous if, and only if, T is effectively continuous. Now apply Theorem 4.6 □

An immediate observation is that, under the conditions of the theorem, a total bounded operator takes computable elements to computable elements, *uniformly*.

Corollary 4.10. *If the linear operator $T: A \to B$ is closed then T is $(\bar{\alpha}, \bar{\beta})$-computable if, and only if, T is bounded. Furthermore, if T is computable then T is total.*

Proof. The subalgebra $\langle e \rangle$ is dense in A and $\langle e \rangle \subseteq \mathrm{dom}(T)$. Then, by the closed graph theorem, T is total in case T is bounded, and hence $\mathrm{dom}(T) \cap \bar{X}^A_{k,\alpha}$ is trivially $\bar{\alpha}$-semicomputable. □

Remark 4.11. Pour-El and Richards prove more than the above for closed linear operators, namely that (under the conditions above) if T is closed and unbounded then there is an $x \in \bar{X}^A_{k,\alpha}$ such that $T(x) \notin \bar{Y}^B_{k,\beta}$. However, this result depends very much on the particular nature of linear operators on Banach spaces and does not generalise to our setting of metric algebras.

We can reduce the dependence on numberings α and β by working with canonical numberings derived from $e: \mathbb{N} \to X$. This is natural in applications where e typically is a basis and $\langle e \rangle$ is a computable Σ-algebra under γ_e.

Theorem 4.12. *Let A and B be Banach spaces with signature Σ. Let $e: \mathbb{N} \to A$ be a partial sequence such that Ω_e is r.e. and $|| \cdot ||: \langle e \rangle \to \mathbb{R}$ is (γ_e, ρ)-computable. Let $T: A \to B$ be a partial linear operator such that Ω_{Te} is r.e., $|| \cdot ||: \langle Te \rangle \to \mathbb{R}$ is (γ_{Te}, ρ)-computable and* $\mathrm{im}(e) \subseteq \mathrm{dom}(T)$*. Then the following are equivalent.*

(i) *T is $(\bar{\gamma}_e, \bar{\gamma}_{he})$-computable; and*

(ii) *T is bounded on $\overline{\langle e \rangle}^A_{k,\gamma_e}$ and* $\mathrm{dom}(T) \cap \overline{\langle e \rangle}^A_{k,\gamma_e}$ *is $\bar{\gamma}_e$-semicomputable.*

Proof. Follows as above from Theorem 4.8. □

Consider the Banach space $C[0,1]$ of continuous functions from the unit interval $[0,1]$ into $\mathbb{R}$ with the uniform norm. The sequence $e(n) = X^n$ generates the subalgebra $\langle e \rangle$ of polynomials with rational coefficients and this algebra is dense in $C[0,1]$. It is rather straight forward to verify that $(\langle e \rangle, \gamma_e)$ is an effective Σ-subalgebra and that the norm on $\langle e \rangle$ is (γ_e, ρ)-computable. Many important partial linear operators are closed and defined on $\mathrm{im}(e)$. Therefore such operators are computable (and hence take computable elements to computable elements) precisely when they are bounded. Thus, for example, integration on $C[0,1]$ is computable while differentiation is not.

4.3.2 Topological groups

Consider the main theorem in the simpler case that the partial algebras A and B are topological groups. A topological group is a group equipped with a topology in which its operations are continuous. Topological groups are used in geometrical investigations, and they occur in topological vector spaces and, hence, in many kinds of function spaces. Each Banach space is also an abelian topological group under addition, and each partial linear operator is also a partial group homomorphism. It turns out that some of the computability-continuity properties of linear operators seen in Section 4.3.1 are true of group homomorphisms in general.

The existence of a continuous group structure has a profound effect on a topological space. For example, it makes the space homogeneous, and continuous homomorphisms are uniformly continuous. The basic facts about topological groups can be found in many books, e.g., Husain [13], Bourbaki [4], Berberian [2] and the survey Comfort [6].

Our Main Theorem 4.6 is about metric algebras so we will focus on metric groups. To fix the notation, we let the signature Σ of a group contain the identity element 1, composition $\cdot$ and inverse $^{-1}$. As usual we write xy for $x{\cdot}y$. A group is a total Σ-algebra. It follows that for a partial homomorphism h between groups, $\mathrm{dom}(h)$ and $\mathrm{im}(h)$ are subgroups.

Definition 4.13. (i) A *metric group* is a topological group whose topology is given by a metric.

(ii) A metric d on a group G is *left invariant* if

$$d(gx, gy) = d(x, y)$$

for all $g, x, y \in G$; the notion of *right invariant* is defined similarly.

Metrics have a natural and important role in computing on a topological group, as indicated by the following fundamental fact:

Theorem 4.14. (Birkhoff-Kakutani) *A topological group G is metrizable if, and only if, it is Hausdorff and the identity element 1_G has a countable fundamental system of neighbourhoods. The metric can be taken to be left or right invariant.*

There exist metric groups that do not possess a metric that is *both* right and left invariant.

Lemma 4.15. *Let G and H be metric groups with left invariant metrics d_G and d_H, respectively. Let $h\colon G \to H$ be a partial group homomorphism. If h is continuous at 1_G then h is uniformly continuous on G. Furthermore, if h is effectively continuous at 1_G then h is effectively uniformly continuous.*

Proof. We show the last statement. Suppose there is a total recursive function $m\colon \mathbb{N} \to \mathbb{N}$ such that for any $x \in \mathrm{dom}(h)$ and $N \in \mathbb{N}$,

$$d_G(1_G, x) < 2^{-m(N)} \implies d_H(1_H, h(x)) < 2^{-N}.$$

It is easy to check that m is a recursive modulus of continuity for all $\mathrm{dom}(h)$. For let $x, y \in \mathrm{dom}(h)$ and assume $d_G(x, y) < 2^{-m(N)}$. Then, by the left invariance of d_G,

$$d_G(x, y) = d_G(x^{-1}x, x^{-1}y) = d_G(1_G, x^{-1}y) < 2^{-m(N)}.$$

By the effective continuity of h at 1_G, we conclude that $d_H(1_H, h(x^{-1}y)) < 2^{-N}$. Since h is a group homomorphism, and d_H is left invariant,

$$d_H(1_H, h(x^{-1}y)) = d_H(1_H, h(x)^{-1}h(y)) = d_H(h(x), h(y)) < 2^{-N}.$$

Note that all the above is defined since $\mathrm{dom}(h)$ and $\mathrm{im}(h)$ are subgroups. Thus, m is a uniform modulus of continuity. □

With this lemma we can deduce an equivalence theorem for metric groups from our Main Theorem 4.6.

Theorem 4.16. *Let G an H be metric groups and assume that (X,α) and (Y,β) are effective metric subgroups of G and H, respectively. Let $e\colon \mathbb{N} \to X$ be a partial α-computable sequence such that $\langle e \rangle$ is a dense subgroup of X.*

Suppose that $h\colon G \to H$ is a partial group homomorphism such that $\mathrm{im}(e) \subseteq \mathrm{dom}(h)$ and $he\colon \mathbb{N} \to Y$ is a partial β-computable sequence. Then the following are equivalent:

(i) *h is $(\bar{\alpha},\bar{\beta})$-computable on $\bar{X}^A_{k,\alpha}$;*

(ii) *h is effectively continuous at the identity 1_G and $\mathrm{dom}(h) \cap \bar{X}^A_{k,\alpha}$ is $\bar{\alpha}$-semicomputable;*

(iii) *h is effectively uniformly continuous on G and $\mathrm{dom}(h) \cap \bar{X}^A_{k,\alpha}$ is $\bar{\alpha}$-semicomputable.*

Consider the additive group $\mathbb{R} = (\mathbb{R}; 0, +, -)$ of real numbers with the standard metric $d(x,y) = |x-y|$, for $x, y \in \mathbb{R}$. Then $\mathbb{R}$ is a metric group. Let (r_n) be a standard recursive enumeration of all rational numbers. Then the sequence $e(n) = r_n$ generates (in fact enumerates) the subgroup $\mathbb{Q}$ of rational numbers. Clearly $(\mathbb{Q}, \gamma_e)$ is a computable group, with a (γ_e, ρ)-computable metric, that is dense in $\mathbb{R}$. The effective closure $\bar{\mathbb{Q}}_{k,\gamma_e}$ is the subgroup $\mathbb{R}_k$ of recursive real numbers, and $\bar{\gamma}_e \sim \rho$.

There is a tremendous range of total homomorphisms $h\colon \mathbb{R} \to \mathbb{R}$. There is a homomorphism $h\colon \mathbb{R} \to \mathbb{R}$ such that for any open interval (a,b) in $\mathbb{R}$, $h|_{(a,b)}\colon (a,b) \to \mathbb{R}$ is surjective (see Comfort [6]). It is easy to check that for any total homomorphism $h\colon \mathbb{R} \to \mathbb{R}$ and all $r \in \mathbb{Q}$,

$$h(r) = h(1) \cdot r.$$

Now suppose that h is ρ-computable on $\mathbb{R}_k$. Then h is continuous on $\mathbb{R}_k$ (since (i) implies (iii)) and, by density of $\mathbb{Q}$ in $\mathbb{R}_k$, we have

$$h(x) = c \cdot x \text{ where } c = h(1)$$

for $x \in \mathbb{R}_k$. Now, a homomorphism $h\colon \mathbb{R} \to \mathbb{R}$ is (ρ,ρ)-computable on $\mathbb{R}_k$ if, and only if, $c \in \mathbb{R}_k$. (This argument works as well for homomorphisms for the additive group $\mathbb{R}^n$ of n-dimensional vectors.)

Constructions with the space $\mathbb{R}$ of real numbers are at the heart of the theory of topological groups. Among the most important topological groups are the classical matrix groups contained in $GL(n,\mathbb{R})$, and abelian groups such as the sphere S^n and torus T^n, all of which are simple constructions with $\mathbb{R}^n$. One may investigate computability using the above methods and results; indeed, for many metric groups G one can find a finite e such that $\langle e \rangle$ is a computable group dense in G.

5 Partial computability structures for metric partial algebras

In this final section we turn to the axiomatic approach of Pour-El and Richards [23]. We generalise their axiomatic approach from Banach spaces to metric partial Σ-algebras. Mori, Tsujii and Yasugi [19, 37] have generalised their axioms to metric spaces, and Washihara [30] to Frechet spaces. A study of representations of topological groups using these methods is Ge and Richards [11].

5.1 Axioms and basic properties of partial computability structures

Given a (real or complex) Banach space X, Pour-El and Richards [23] provided axioms for a *computability structure* on X. A computability structure on X is a set of total sequences on X satisfying their axioms.

A Banach space is a two sorted metric algebra, which can be viewed as a single sorted metric Σ-algebra as described in Section 4.3.1. Here we generalise their axioms to metric partial Σ-algebras. In particular it is natural, a priori, to consider partial sequences.

Before stating the axioms we need to make precise what we mean by a converging partial sequence on a metric space (A, d_A).

Definition 5.1.

(i) Let $x\colon \mathbb{N} \to A$ be a partial sequence and let $y \in A$. Then x *converges to* y (denoted $x(n) \to y$) if $\mathrm{dom}(x)$ is infinite and

$$\forall N \exists m (n \geq m \wedge x(n)\downarrow \implies d_A(x(n), y) < 2^{-N}).$$

(ii) An *effective modulus* for a partial sequence $x\colon \mathbb{N} \to A$ converging to $y \in A$ is a total recursive increasing function m such that

$$(\forall n \geq m(N))(x(n)\downarrow \implies d_A(x(n), y) < 2^{-N}).$$

Definition 5.2. Let A be a metric partial Σ-algebra and let C_A be a non-empty set of partial sequences on A. Then C_A is a *partial computability structure* on A if the four axioms below hold.

Axiom 1 (Term evaluation)
If $x \in C_A$ and (t_n) is a partial γ-computable sequence of terms from $T(\Sigma, V)$ then $(t_n^A(x)) \in C_A$.

Axiom 2 (Double sequence)
Let $(x_{nk}) \in C_A$ be a partial double sequence. Suppose W is an r.e. set and $m(n, N)$ a partial recursive function such that

$$n \in W \implies x_{nk} \to x_n \in A \text{ with effective modulus } \lambda N.m(n, N).$$

Then the partial sequence (x_n) is in C_A.

Axiom 3 (Metric)
If $(x_n), (y_n) \in C_A$ then $(d_A(x_n, y_m))$ is a ρ-computable partial double sequence in $\mathbb{R}$.

Axiom 4 (Merge)
If $(x_n), (y_n) \in C_A$ then $(x_0, y_0, x_1, y_1, \ldots) \in C_A$.

Axiom 3 is a key axiom here, in that it tightly links the axiomatic approach to the approach via numberings, as we shall show in the next section. This is also true in Pour-El and Richards Banach space axiomatisation. As a simple illustration we have the following lemma.

Lemma 5.3. *Let C_A be a computability structure over A. Then* $\mathrm{dom}(x)$ *is r.e. whenever $x \in C_A$.*

Proof. The partial sequence $(d_A(x(n), x(n)))$ is a partial ρ-computable sequence in $\mathbb{R}$ by Axiom 3. It follows that $\mathrm{dom}(x)$ is an r.e. set. □

Axiom 4 is provable in the usual situation of a computable generating sequence. However, the axiom is necessary in order to prove the equivalence of our formulation with that of Pour-El and Richards for Banach spaces.

For real Banach spaces we consider, as usual, the signature $\Sigma = \{0, +, r : r \in \mathbb{Q}\}$. In the following proposition we restrict Axiom 2 by only allowing $W = \omega$, which is no loss of generality by Lemma 5.3.

Proposition 5.4. *Let X be a real Banach space and let C_X be a set of total sequences of X. Then C_X is a computability structure on X in the sense of Pour-El and Richards if, and only if, it is a computability structure in the sense above with respect to Σ.*

We refer to [23] for the exact formulation of the Pour-El and Richards axiomatisation. The proof is rather straight forward noting that the Pour-El and Richards Axiom 1 can be obtained using Axiom 4, Axiom 1 and Axiom 2, and effective Cauchy sequences of rationals for the recursive reals involved.

Examples

(i) Consider the field $\mathbb{R}$ as a metric partial algebra with constants 0 and 1 and the total operations of addition, subtraction and multiplication, and the partial inverse operation. Let $C_{\mathbb{R}}$ be the set of partial ρ-computable sequences on $\mathbb{R}$. It is easily verified that $C_{\mathbb{R}}$ is a computability structure on $\mathbb{R}$.

(ii) Let (X, α) be an effective metric space with a limit algorithm. Let C_X be the set of partial α-computable sequences. Then C_X is a computability structure on X.

(iii) Let A be a metric space and $X \subseteq A$. Suppose (X, α) is an effective metric space. Let $C_{\bar{X}^A_{k,\alpha}}$ be the set of all partial $\bar{\alpha}$-computable sequences on $\bar{X}^A_{k,\alpha}$. Then $C_{\bar{X}^A_{k,\alpha}}$ is a computability structure on A. Axiom 2 holds since $(\bar{X}^A_{k,\alpha}, \bar{\alpha})$ has a limit algorithm.

From Axiom 1, independently of Σ, we see that C_A is closed under partial recursive permutations.

Lemma 5.5. *Let C_A be a computability structure over A and let $r\colon \mathbb{N} \to \mathbb{N}$ be a partial recursive function. If $(x_n) \in C_A$ then $(x_{r(n)}) \in C_A$.*

Proof. Let $x = (x_n) \in C_A$ and consider the partial γ-computable sequence $(v_{r(n)})$ in $T(\Sigma, V)$. Then

$$(x_{r(n)}) = (v_{r(n)}(x)) \in C_A$$

by Axiom 1. □

We now present a computability structure in the setting of the previous sections.

Proposition 5.6. *Let A be a metric partial Σ-algebra and assume (X, α) is an effective metric partial Σ-algebra such that X is a metric partial Σ-subalgebra of A. Suppose $e\colon \mathbb{N} \to X$ is a partial α-computable sequence such that $\langle e \rangle$ is dense in X. Suppose further that $(\bar{X}^A_{k,\alpha}, \bar{\alpha})$ is an effective partial Σ-algebra. Let $C_{\bar{X}^A_{k,\alpha}}$ be the set of partial $\bar{\alpha}$-computable sequences. Then $C_{\bar{X}^A_{k,\alpha}}$ is a computability structure on the metric partial Σ-algebra A.*

Proof. Axiom 1 holds since we have assumed that $(\bar{X}^A_{k,\alpha}, \bar{\alpha})$ is an effective Σ-algebra. Axiom 2 holds since $(\bar{X}^A_{k,\alpha}, \bar{\alpha})$ has a limit algorithm by Lemma 3.4. Axiom 3 holds since $(\bar{X}^A_{k,\alpha}, \bar{\alpha})$ is an effective metric space, and Axiom 4 trivially holds. □

Observe that the hypothesis that $(\bar{X}^A_{k,\alpha}, \bar{\alpha})$ is an effective partial Σ-algebra is necessary in the proof above but that this hypothesis does not appear in Theorem 4.6. Thus our main theorem is more general than the corresponding theorem below for computability structures.

5.2 Models and the representation theorem

In this section we show that every computability structure is (essentially) characterised by Proposition 5.6. The key property is the Effective Density Lemma. First we have the following observation.

Lemma 5.7. *Let A be a metric partial Σ-algebra and let C_A be a computability structure over A. Suppose $e\colon \mathbb{N} \to A$ is a partial sequence such that $e \in C_A$. Then $(\langle e \rangle, \gamma_e)$ is an effective metric partial Σ-algebra. Furthermore there is a partial γ_e-computable sequence $(p_n) \in C_A$ enumerating all of $\langle e \rangle$.*

Proof. Define p_n by $p_n \simeq \mathrm{TE}(\gamma(n), e) \simeq \gamma_e(n)$. Then $\langle e \rangle = \{p_n : p_n{\downarrow} \wedge n \in \mathbb{N}\}$. It follows from Axiom 1 that $(p_n) \in C_A$ and hence, by Lemma 5.3, that Ω_e is r.e. Then $(\langle e \rangle, \gamma_e)$ is an effective partial Σ-algebra by Proposition 2.17. Furthermore (p_n) is a γ_e-computable partial sequence, again since Ω_e is r.e., tracked by the identity function. Axiom 3 asserts that $(d_A(p_n, p_m))$ is a partial ρ-computable double sequence. This implies that $d_A|_{\langle e \rangle^2}$ is (γ_e, ρ)-computable, i.e., $(\langle e \rangle, \gamma_e)$ is effective as a metric space. $\square$

Lemma 5.8. (Effective Density Lemma) *Let A be a metric partial Σ-algebra with a computability structure C_A. Suppose $e\colon \mathbb{N} \to A$ is a partial sequence such that $e \in C_A$ and $\langle e \rangle$ is dense in A. Then the following are equivalent.*

(i) $(x_n) \in C_A$.

(ii) *There is a partial double sequence $(p_{nk}) \in C_A$ such that*

- $(p_{nk}) \to (x_n)$ *effectively (as in Axiom 2);*
- (p_{nk}) *is a partial γ_e-computable double sequence.*

Proof. (ii) $\implies$ (i) is just Axiom 2. We now prove the converse. Let (p_n) be the γ_e-computable sequence from Lemma 5.7 enumerating $\langle e \rangle$. Suppose we are given $(x_n) \in C_A$. Axiom 3 guarantees that $(d_A(x_n, p_m))$ is a partial ρ-computable double sequence. Thus we may define a partial recursive function $i(n, k)$ by

$$i(n,k) \simeq \text{some } m[x_n{\downarrow} \wedge p_m{\downarrow} \wedge d_A(x_n, p_m) < 2^{-k}].$$

Note that $i(n,k){\downarrow} \iff x_n{\downarrow}$ since $\langle e \rangle$ is dense in A. Now set $p_{nk} \simeq p_{i(n,k)}$. Then (p_{nk}) is a partial γ_e-computable double sequence and $(p_{nk}) \in C_A$ by Lemma 5.5. Let $W = \{n : x_n{\downarrow}\}$. Then W is r.e. and $\lambda k.p_{nk}$ is total for $n \in W$. Clearly the function $m(n, N) = N$ is an effective modulus showing that $(p_{nk}) \to (x_n)$ effectively. $\square$

The stability theorem is a corollary of the effective density lemma.

Corollary 5.9. *Let A be a metric partial Σ-algebra and let $e\colon \mathbb{N} \to A$ be a partial sequence such that $\langle e \rangle$ is dense in A. Suppose C_A and C'_A are computability structures over A such that $e \in C_A$ and $e \in C'_A$. Then $C_A = C'_A$.*

Proof. Let (p_n) be the γ_e-computable sequence from Lemma 5.7 enumerating $\langle e \rangle$. Note that (p_n) is in C_A and in C'_A by Axiom 1. Suppose $(x_n) \in C_A$. Then by the Effective Density Lemma 5.8, there is a partial double sequence (p_{nk}), a partial recursive permutation of (p_n), such that

$$(p_{nk}) \to (x_n)$$

effectively. But $(p_{nk}) \in C'_A$ by Lemma 5.5, and hence $(x_n) \in C'_A$ by Axiom 2. □

Theorem 5.10. *Let A be a metric partial Σ-algebra and let $e: \mathbb{N} \to A$ be a partial sequence such that $\langle e \rangle$ is dense in A. Suppose C_A is a computability structure over A and that $e \in C_A$. Then*

$$(x_n) \in C_A \iff (x_n) \text{ is } \bar{\gamma}_e\text{-computable.}$$

Proof. Assume $(x_n) \in C_A$. The proof of the effective density lemma is uniform. Thus from the partial ρ-computable double sequence $(d_A(x_n, p_m))$ we obtain a partial γ_e-computable double sequence (p_{nk}) and a recursive modulus function such that

$$(p_{nk}) \to (x_n)$$

as in Axiom 2. Note that when $x_n\downarrow$ then $x_n \in \overline{\langle e \rangle}^A_{k,\gamma_e}$. Using the $(\gamma_e, \bar{\gamma}_e)$-computable embedding of $\langle e \rangle$ into $\overline{\langle e \rangle}^A_{k,\gamma_e}$ and the $\bar{\gamma}_e$-computable limit algorithm we see that (x_n) is $\bar{\gamma}_e$-computable.

For the converse we first note that any partial recursive permutation $(p_{r(n)})$ of (p_n) is in C_A by Lemma 5.7 and Lemma 5.5, and hence every γ_e-computable sequence is in C_A. Let (x_n) be a partial $\bar{\gamma}_e$-computable sequence. Then there is a γ_e-computable partial double sequence which converges to (x_n) effectively. Axiom 2 now guarantees that $(x_n) \in C_A$. □

Here is a generalisation of the first main theorem of Pour-El and Richards to partial computability structures over metric partial Σ-algebras.

Theorem 5.11. *Let A and B be partial metric Σ-algebras and let C_A and C_B be computability structures on A and B, respectively. Let $e: \mathbb{N} \to A$ be a partial sequence such that $e \in C_A$.*

Suppose $h: A \to B$ is a partial Σ-homomorphism such that $he \in C_B$ and h is a strong partial Σ-homomorphism on $\langle e \rangle$. Then $h|_{\overline{\langle e \rangle}^A_{k,\gamma_e}}$ is $(\bar{\gamma}_e, \bar{\gamma}_{he})$-computable if, and only if, $h|_{\overline{\langle e \rangle}^A_{k,\gamma_e}}$ is effectively $\bar{\gamma}_e$-continuous and its domain $\mathrm{dom}(h|_{\overline{\langle e \rangle}^A_{k,\gamma_e}})$ is $\bar{\gamma}_e$-semicomputable.

Proof. From Lemmas 5.3 and 5.7 we see that the hypotheses of Theorem 4.8 are satisfied. □

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