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Model Companions of T_σ for stable T *

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Let T be a complete first order theory in a countable relational language L. We assume relation symbols have been added to make each formula equivalent to a predicate. Adjoin a new unary function symbol σ to obtain the language L_σ; T_σ is obtained by adding axioms asserting that σ is an L-automorphism.

The modern study of the model companion of theories with an automorphism has two aspects. One line, stemming from Lascar [6], deals with 'generic' automorphisms of arbitrary structures. A second, beginning with Chatzidakis and Hrushovski [1] and questions of Macintyre about the Frobenius automorphism is more concerned with specific algebraic theories. This paper is more in the first tradition: we find general necessary and sufficient conditions for a stable first order theory with automorphism to have a model companion.

Kikyo investigates the existence of model companions of T_σ when T is unstable in [3]. He also includes an argument of Kudaibergenov showing that

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if T is stable with the finite cover property then T_σ has no model companion. This argument was implicit in [2] and is a rediscovery of a theorem of Winkler [9] in the 70's. We provide necessary and sufficient conditions for T_σ to have a model companion when T is stable. Namely, we introduce a new condition: T_σ admits obstructions, and show that T_σ has a model companion iff and only if T_σ does not admit obstructions. This condition is weakening of the finite cover property: if a stable theory T has the finite cover property then T_σ admits obstructions.

Kikyo also proved that if T is an unstable theory without the independence property, T_σ does not have a model-companion. Kikyo and Shelah [5] have improved this by weakening the hypothesis to, T has the strict order property.

For p a type over A and σ an automorphism with $A \subset \mathrm{dom}\ \mathrm{p}$, $\sigma(p)$ denotes $\{\phi(\mathbf{x}, \sigma(\mathbf{a})) : \phi(\mathbf{x}, \mathbf{a}) \in p\}$. References of the form II.4.13. are to [8]

1 Example

In the following example we examine exactly why a particular T_σ does not have a model companion. Eventually, we will show that the obstruction illustrated here represents the reason T_σ (for stable T) can fail to have a model companion. Let L contain two binary relation symbols E and R and unary predicates P_i for $i < \omega$. The theory T asserts that E is an equivalence relation with infinitely many infinite classes, which are refined by R into two-element classes. Moreover, each P_i holds only of elements from one E-class and contains exactly one element from each R class of that E-class.

Now, T_σ does not have a model companion. To see this, let $\psi(x, y, z)$ be the formula: $E(x, z) \wedge E(y, z) \wedge R(x, y) \wedge x \neq y$. Let Γ be the L_σ-type in the variables $\{z\} \cup \langle x_i y_i : i < \omega\rangle$ which asserts for each i, $\{\psi(x_i, y_i, z)\}$, the sequence $\langle x_i y_i : i < \omega\rangle$ is L-indiscernible and for every $\phi(x, \mathbf{w}) \in L(T)$:

$$(\forall \mathbf{w}) \bigvee \{ \bigwedge_{i \in U} \phi(x_i, \mathbf{w}) \leftrightarrow \phi(y_i, \sigma(\mathbf{w})) : U \subseteq \lg(\mathbf{w}) + 3, |U| > (\lg(\mathbf{w}) + 3)/2 \}.$$

Thus if $\langle b_i c_i : i < \omega\rangle a$ realize Γ in a model M,

$$\sigma(\mathrm{avg}(\langle \mathrm{b_i} : \mathrm{i} < \omega\rangle/\mathrm{M})) = \mathrm{avg}(\langle \mathrm{c_i} : \mathrm{i} < \omega\rangle/\mathrm{M}).$$

For any finite $\Delta \subset L(T)$, let $\chi_{\Delta,k}(\mathbf{x}, \mathbf{y}, z)$ be the conjunction of the Δ-formulas satisfied by $\langle b_i c_i : i < k\rangle a$ where $\langle b_i c_i : i < k\rangle a$ are an initial segment of a realization of Γ. Let $\theta_{\Delta,k}$ be the sentence

$$(\forall x_0,\dots x_{k-1},y_0,\dots y_{k-1},z)\chi_{\Delta,k}(x_0,\dots x_{k-1},y_0,\dots y_{k-1},z)\rightarrow$$

$$(\exists x_0,y_0,x_1,y_1)[\psi(x_0,y_0,z)\wedge\psi(x_1,y_1,z)\wedge\sigma(x_1)=y_1].$$

We now claim that if T_σ has a model companion T^*_σ, then for some k and Δ,

$$T^*_\sigma \vdash \theta_{\Delta,k}.$$

For this, let $M \models T^*_\sigma$ such that $\langle b_i c_i : i < k\rangle a$ satisfy Γ in M. Suppose $M \mid L \prec N$ and N is an $|M|^+$-saturated model of T. In N we can find b, c realizing the average of $\langle b_i : i < \omega\rangle$ and $\langle c_i : i < \omega\rangle$ over M respectively. Then

$$\sigma(\mathrm{avg}(\langle \mathrm{b_i} : \mathrm{i} < \omega\rangle/\mathrm{M})) = \mathrm{avg}(\langle \mathrm{c_i} : \mathrm{i} < \omega\rangle/\mathrm{M})$$

and so there is an automorphism σ^* of N extending σ and taking b to c. Since (M,σ) is existentially closed (T^*_σ is model complete), we can pull b, c down to M. By compactness, some finite subset Γ_0 of Γ suffices and letting Δ be the formulas mentioned in Γ_0 and k the number of x_i, y_i appearing in Γ_0 we have the claim.

But now we show that if (M,σ) is any model of T_σ, then for any finite Δ and any k, $(M,\sigma) \models \neg\theta_{\Delta,k}$. For this, choose b_i, c_i for $i < k$ which are E-equivalent to each other and to an element a in a class P_j where P_j does not occur in Δ and with $R(b_i, c_i)$ and $b_i \neq c_i$. Then $\mathbf{b}, \mathbf{c}, a$ satisfy $\chi_{\Delta,k}$ but there are no b_k, c_k and automorphism σ which makes $\theta_{\Delta,k}$ true. For, for each j,

$$T \vdash (\forall x,y,z)(\psi(x,y,z)\wedge P_j(z)\rightarrow [P_j(x)\leftrightarrow \neg P_j(y)]).$$

To put this situation in a more general framework, recall some notation from [8]. Δ will note a finite set of formulas: $\{\phi_i(\mathbf{x},\mathbf{y}_i) : \mathrm{lg}(\mathbf{x}) = m, i < |\Delta|\}$; p is a Δ-m-type over A if p is a set of formulas $\phi_i(\mathbf{x},\mathbf{a})$ where $\mathbf{x} = \langle x_1,\dots x_{m-1}\rangle$ (these specific variables) and $\mathbf{a}$ from A is substituted for $\mathbf{y}_i$. Thus, if A is finite there are only finitely many Δ-m-types over A.

Now let Δ_1 contain Boolean combinations of $x = y, R(x,y), E(x,y)$. Let Δ_2 expand Δ_1 by adding a finite number of the $P_j(z)$ and let Δ_3 contain $P_j(x)$ where P_j does not occur in Δ_2.

Now we have the following situation: there exists a set $X = \{b_0, b_1, c_0, c_1, a\}$, $P_j(a)$ holds, all 5 are E-equivalent and $R(b_i, c_i)$ for $i = 0, 1$ such that:

1. $\langle \mathbf{b}_i\mathbf{c}_i : i \leq 2\rangle$ is Δ_2-indiscernible over a.

2. $\langle b_0c_0, b_1c_1\rangle$ can be extended to an infinite set of indiscernibles $\overline{\mathbf{b}\mathbf{c}}$ which satisfy the following.

3. $\psi(b_i, c_i, a)$.

4. $\sigma(\mathrm{avg}_{\Delta_2}(\overline{\mathbf{b}}/\mathrm{M}) = \mathrm{avg}_{\Delta_2}(\overline{\mathbf{c}}/\mathrm{M})$.

5. $\mathrm{tp}_{\Delta_1}(\mathbf{b}_2\mathbf{c}_2/\mathrm{X}) \vdash \sigma(\mathrm{tp}_{\Delta_3}(\mathbf{b}_2/\mathrm{X})) \neq \mathrm{tp}_{\Delta_3}(\mathbf{c}_2/\mathrm{X})$.

We call a sequence like $\langle \mathbf{b}_i\mathbf{c}_i : i \leq 2\rangle\mathbf{a}$ a $(\sigma, \Delta_1, \Delta_2, \Delta_3, n)$-obstruction over the emptyset. In order to 'finitize' the notions we will give below more technical formulations of the last two conditions: we will have to discuss obstructions over a finite set A. In the example, the identity was the only automorphism of the prime model. We will have to introduce a third sequence $\mathbf{b}'$ to deal with arbitrary σ. But this example demonstrates the key aspects of obstruction which are the second reason for T_σ to lack a model companion.

2 Preliminaries

In order to express the notions described in the example, we need several notions from basic stability theory. By working with finite sets of formulas in a stable theory without the finite cover property we are able to refine arguments about infinite sets of indiscernibles to arguments about sufficiently long finite sequences. Let Δ be a finite set of formulas which we will assume to be closed under permutations of variables and negation; $\neg\neg\phi$ is identified with ϕ. Recall that an ordered sequence $E = \langle \mathbf{a}_i : i \in I\rangle$ is said to be (Δ, p) indiscernible over A if any two properly ordered p-element subsequences of I realize the same Δ-type over A. It is Δ-indiscernible if it is (Δ, p)-indiscernible for all p, or equivalently for all p' with p' at most the maximum number of variables in a formula in Δ. For any sequence $E = \langle \mathbf{a}_i : i \in I\rangle$ and $j \in I$ we write E_j for $\langle \mathbf{a}_i : i < j\rangle$.

We will rely on the following facts/definitions from [8] to introduce two crucial functions for this paper: $F(\Delta, n)$ and $f(\Delta, n)$.

Fact 2.1 *Recall that if T is stable, then for every finite $\Delta \subset L(T)$ and $n < \omega$ there is a finite $\Delta' = F(\Delta, n)$ with $\Delta \subseteq \Delta' \subset L(T)$ and a $k^* = f(\Delta, n)$ such that*

1. *A sequence* $\langle \mathbf{e}_i : i \in I\rangle$ *of* n*-tuples such that for* $i < j$ *and a finite set* A, $\mathrm{tp}_{F(\Delta,n)}(\mathrm{e_j/E_iA}) = \mathrm{tp}_{F(\Delta,n)}(\mathrm{e_i/E_iA})$ *and* $R(F(\Delta, n), 2)(e_j/E_jA) = R(F(\Delta, n), 2)(e_i/E_iA)$, *(whence,* $\mathrm{tp}_{F(\Delta,n)}(\mathrm{e_j/E_iA})$ *is definable over* A*) is a sequence of* Δ*-indiscernibles (II.2.17).*

2. *For any set of* Δ'*-indiscernibles over the empty set,* $E = \langle \mathbf{e}_i : i < k\rangle$ *with* $\lg(\mathbf{e}_i) = n$ *and* $k \geq k^*$ *for any* $\theta(\mathbf{u}, \mathbf{v}) \in \Delta$ *and any* $\mathbf{d}$ *with* $\lg(\mathbf{d}) = \lg(\mathbf{v}) = m$ *either* $\{\mathbf{e}_i : \phi(\mathbf{e}_i, \mathbf{d})\}$ *or* $\{\mathbf{e}_i : \neg\phi(\mathbf{e}_i, \mathbf{d})\}$ *has strictly less than* $k^*/2$ *elements. (II.4.13., II.2.20)*

3. *This implies that, for appropriate choice of* k^*,

   (a) *there is an integer* $m = m(\Delta, n) \geq n$ *such that for any set of* Δ'*-indiscernibles* $\langle \mathbf{e}_i : i < k\rangle$ *over* A *with* $\lg(\mathbf{e}_i) = n$ *and* $k \geq k^*$ *and any* $\mathbf{a}$ *with* $\lg(\mathbf{a}) \leq m$ *there is a* $U \subseteq k$ *with* $|U| < k^*/2$ *such that* $\langle \mathbf{e}_i : i \in k - U\rangle$ *is* Δ*-indiscernible over* $A\mathbf{a}$*;*

   (b) *moreover if* $k \geq k^*$*, for any set* A, $\mathrm{avg}_\Delta(\langle \mathbf{e}_i : i < k\rangle/A)$ *is well-defined. Namely,* $\mathrm{avg}_\Delta(\langle \mathbf{e}_i : i < k\rangle/A) =$

$$\{\phi(\mathbf{x}, \mathbf{a}) : |\{\mathbf{e}_i : i < k, \phi(\mathbf{e}_i, \mathbf{a})| \geq \frac{k^*}{10}, \mathbf{a} \in A, \phi(\mathbf{x}, \mathbf{y}) \in \Delta\}.$$

In a), m is the least $k \geq n$ such all $\phi \in \Delta$ have at most k free variables. But $\mathrm{avg}_\Delta(\langle \mathbf{e}_i : i < k\rangle/A)$ need not be consistent. (Let E be all the members of one finite class in the standard fcp example and let $A = E$.) However,

Fact 2.2 *If, in addition to Fact 2.1,* T *does not have the finite cover property, we can further demand*

1. *If* $E = \langle \mathbf{e}_i : i < k^*\rangle$ *is a set of* Δ'*-indiscernibles over the empty set, for any* A, $\mathrm{avg}_\Delta(E/A)$ *is a* consistent *complete* Δ*-type over* A.

2. *Moreover* k^* *can be chosen so that any set of* Δ'*-indiscernibles (of* n*-tuples) with length at least* k^* *can be extended to one of infinite length (II.4.6).*

3. *For any pair of* $F(F(\Delta, n))$*-indiscernible sequences* $E^1 = \langle \mathbf{e}^1_i : i < k\rangle$ *and* $E^2 = \langle \mathbf{e}^2_i : i < k\rangle$ *over* $\mathbf{a}$ *with* $\lg(\mathbf{e}_i) = n$ *and* $k \geq k^*$ *such that*

$$\mathrm{avg}_{F(\Delta)})(\mathrm{E}^1/\mathbf{a}\mathrm{E}^1\mathrm{E}^2) = \mathrm{avg}_{F(\Delta)})(\mathrm{E}^2/\mathbf{a}\mathrm{E}^1\mathrm{E}^2),$$

there exist $\mathbf{J} = \langle \mathbf{e}_j : k < j < \omega \rangle$ *such that both* $E^1 J$ *and* $E^2 J$ *are* $F(\Delta)$-*indiscernible over* $\mathbf{a}$. *We express the displayed condition on* $\overline{\mathbf{e}}^1, \overline{\mathbf{e}}^2$ *by the formula:* $\lambda_\Delta(\overline{\mathbf{e}}^1, \overline{\mathbf{e}}^2, \mathbf{a})$.

4. *If* E_1 *and* E_2 *contained in a model* M *are* $F(\Delta, n)$-*indiscernible over* $\mathbf{a} \in M$ *and each have length at least* k^*, *there is a formula* $\lambda_\Delta(\mathbf{x}^1, \mathbf{x}^2, \mathbf{z})$ *such that* $M \models \lambda_\Delta(\mathbf{e}^1, \mathbf{e}^2, \mathbf{a})$ *if and only if* $\mathrm{avg}_\Delta(\mathbf{e}^1/\mathrm{M}) = \mathrm{avg}_\Delta(\mathbf{e}_2/\mathrm{M})$.

Proof. For 1, make sure that k^* is large enough that every Δ-type which is k^*-consistent is consistent (II.4.4 3)). Now 3) follows by extending the common $F(\Delta)$-average of E^1 and E^2 over $\mathbf{a}E^1E^2$ by 2). Finally, condition 4 holds by adapting the argument for III.1.8 from the set of all L-formulas to Δ; λ_Δ is the formula from 3).

Note that both F and f can be chosen increasing in Δ and n.

3 Obstructions

In this section we introduce the main new notion of this paper: obstruction.

We are concerned with a formula $\psi(\mathbf{x}, \mathbf{y}, \mathbf{z})$ where $\lg(\mathbf{x}) = \lg(\mathbf{y}) = n$ and $\lg(\mathbf{z}) = m$. We will apply Facts 2.1 and 2.1 with $\mathbf{e}_i = \mathbf{b}_i \mathbf{b}'_i \mathbf{c}_i$ where each of $\mathbf{b}_i$, $\mathbf{b}'_i$, and $\mathbf{c}_i$ has length n. Thus, our exposition will depend on functions $F(\Delta, 3n)$, $f(\Delta, 3n)$. In several cases, we apply Fact 2.1 with $\phi(\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{v})$ as $\theta(\mathbf{u}_2, \mathbf{v}) \leftrightarrow \theta(\mathbf{u}_3, \mathbf{v})$ for various θ.

The following notation is crucial to state the definition.

Notation 3.1 *If* $\overline{\mathbf{d}} = \langle \mathbf{d}_i : i < r \rangle$ *is a sequence of* $3n$-*tuples, which is* Δ-*indiscernible over a finite sequence* $\mathbf{f}$, *and* $r \geq k^* = f(\Delta, 3n)$, *then* $\tau_\Delta(\mathbf{z}, \overline{\mathbf{d}}\mathbf{f})$ *is the formula with free variable* $\mathbf{z}$ *and parameters* $\overline{\mathbf{d}}\mathbf{f}$ *which asserts that there is a subsequence* $\overline{\mathbf{d}}'$ *of* $\overline{\mathbf{d}}$ *with length* $f(\Delta, 3n)$ *so that* $\overline{\mathbf{d}}'\mathbf{z}$ *forms a sequence of* Δ-*indiscernibles over* $\mathbf{f}$.

The following result follows easily from Fact 2.2 3) and the definition of $f(\Delta, n)$.

Lemma 3.2 *Let* $p \geq k = f(\Delta, n)$ *and suppose* $\langle \mathbf{e}_i : i < p \rangle$ *is a sequence of* $3n$-*tuples which is* $F(\Delta, 3n)$- *indiscernible over* $\mathbf{f}$. *If* $S_1, S_2 \subseteq p$ *with* $|S_1|, |S_2| \geq k^*/2$ *then* $\tau_\Delta(\mathbf{z}, \overline{\mathbf{d}}|S_1\mathbf{f})$ *and* $\tau_\Delta(\mathbf{z}, \overline{\mathbf{d}}|S_2\mathbf{f})$ *are equivalent formulas.*

Now we come to the main notion. Intuitively, $\langle \mathbf{b}_i \mathbf{b}'_i \mathbf{c}_i : i \leq k\rangle \mathbf{a}$ is a $(\Delta_1, \Delta_2, \Delta_3, n)$ obstruction over A if $\langle \mathbf{b}_i \mathbf{b}'_i \mathbf{c}_i : i \leq k\rangle$ is an indefinitely extendible sequence of Δ_2 indiscernibles over $\mathbf{a}$ such that the $\mathbf{b}$'s, $\mathbf{b}'_i$'s and $\mathbf{c}$'s each have length n and the Δ_2-average of the $\mathbf{b}'_i$ and $\mathbf{c}$ is the same (over any set) but any realizations of the Δ_1-type of the $\mathbf{b}'_i$ and the Δ_1-type of the $\mathbf{c}_i$ over $\mathbf{a}$ and the sequence have different Δ_3-types over A. More formally, we define:

Definition 3.3 *For finite* $\Delta_1 \subseteq \Delta_2 \subseteq L(T)$ *and* $\Delta_3 \subseteq L(T)$, *finite* $\mathbf{a} \subseteq A \subset M \models T$ *with* $\lg(\mathbf{a}) \leq m(\Delta, n)$ *(as in Fact 2.1),* σ *an automorphism of* M, *and a natural number* n, $\langle \mathbf{b}_i \sigma(\mathbf{b}_i) \mathbf{c}_i : i \leq k\rangle \mathbf{a}$ *is a* $(\sigma, \Delta_1, \Delta_2, \Delta_3, n)$ obstruction over A *if the following conditions hold.*

1. $\langle \mathbf{b}_i \sigma(\mathbf{b}_i) \mathbf{c}_i : i \leq k\rangle$ *is* $F(\Delta_2, 3n)$*-indiscernible over* $\mathbf{a}$.
2. $k \geq f(\Delta_2, 3n)$.
3. $\mathrm{avg}_{\Delta_2}(\overline{\mathbf{e}}^1/\mathrm{M}) = \mathrm{avg}_{\Delta_2}(\overline{\mathbf{e}}^2/\mathrm{M})$ *where* $(\overline{\mathbf{e}}^1 = \langle \sigma(\mathbf{b}_i) : i < k\rangle$ *and* $\overline{\mathbf{e}}^2 = \langle \mathbf{c}_i : i < k\rangle$.
4. *Writing* $\tau_{\Delta_1}(\langle \mathbf{b}_i \sigma(\mathbf{b}_i) \mathbf{c}_i) : i < k\rangle \mathbf{a})$ *with free variables* $\mathbf{x}, \mathbf{x}', \mathbf{y}$, *we have*

$$M \models (\forall \mathbf{x}, \mathbf{x}', \mathbf{y})[\tau_{\Delta_1}(\mathbf{x}, \mathbf{x}', \mathbf{y}, \langle \mathbf{b}_i \sigma(\mathbf{b}_i) \mathbf{c}_i : i < k\rangle \mathbf{a}) \rightarrow$$

$$\bigvee \{\phi(\mathbf{x}', \mathbf{f}) \wedge \neg\phi(\mathbf{y}, \mathbf{f}) : \mathbf{f} \in A \cup \bigcup_{i<k} \langle \mathbf{b}_i \sigma(\mathbf{b}_i) \mathbf{c}_i\rangle \cup \mathbf{a}, \phi \in \Delta_3\}]$$

By Fact 2.2, Condition 3) is expressed by a formula of $\mathbf{e}^1, \mathbf{e}^2$ and $\mathbf{a}$. Crucially, the hypothesis of the fourth condition in Definition 3.3 is an L-formula with parameters $\langle \mathbf{b}_i \sigma(\mathbf{b}_i) \mathbf{c}_i) : i < k\rangle \mathbf{a}$; the conclusion is an L-formula with parameters from A as well. The disjunction in the conclusion of condition 4) is nonempty since each $\mathbf{b}_i$, $\sigma(\mathbf{b}_i)$ is in the domain if Δ_3 is nontrivial. Δ_1 and Δ_2 have $3n$ type-variables; Δ_3 has n type-variables.

Fact 3.4 *Note that if* $\langle \mathbf{b}_i \mathbf{c}_i : i \leq k\rangle \mathbf{a}$ *is a* $(\sigma, \Delta_1, \Delta_2, \Delta_3, n)$ *obstruction over* A *and* $\Delta_1 \subseteq \Delta'_2 \subseteq \Delta_2$, *then* $\langle \mathbf{b}_i \mathbf{c}_i : i \leq k\rangle \mathbf{a}$ *is a* $(\sigma, \Delta_1, \Delta'_2, \Delta_3, n)$ *obstruction over* A. *Further, if* $\langle \mathbf{b}_i \mathbf{c}_i : i \leq k\rangle \mathbf{a}$ *is a* $(\sigma, \Delta_1, \Delta_2, \Delta_3, n)$ *obstruction over* A *and* $A \subseteq A'$, *where* A' *is finite,* $\Delta_1 \subseteq \Delta'_2 \subseteq \Delta_2$, *then* $\langle \mathbf{b}_i \mathbf{c}_i : i \leq k\rangle \mathbf{a}$ *is a* $(\sigma, \Delta_1, \Delta'_2, \Delta_3, n)$ *obstruction over* A'.

Definition 3.5 *1. We say* $(M,\sigma) \models T_\sigma$ has no σ-obstructions *when there is a function* $G(\Delta_1, n)$ *with* $F(\Delta_1, 3n) \subseteq G(\Delta_1, n)$ *such that if* Δ_1 *is a finite subset of* $L(T)$ *and* $G(\Delta_1, n)$ *is contained in the finite* $\Delta_3 \subset L(T)$*, then for every finite subset* A *of* M*, there is no* $(\sigma, \Delta_1, G(\Delta_1, n), \Delta_3, n)$ *obstruction over* A*.*

2. We say T_σ has no σ-obstructions *when there is a function* $G(\Delta_1, n)$ *(which does not depend on* (M,σ)*) such that for each* $(M,\sigma) \models T_\sigma$*, if* Δ_1 *is a finite subset of* $L(T)$*,* A *is finite subset of* M*, and* Δ_3 *is a finite subset of* $L(T)$*, there is no* $(\sigma, \Delta_1, G(\Delta_1, n), \Delta_3, n)$ *obstruction over* A*.*

Definition 3.6 *A simple obstruction is an obstruction where the automorphism* σ *is the identity. The notions of a theory or model having a simple obstruction are the obvious modifications of the previous definition.*

Lemma 3.7 T_σ *has obstructions if and only if* T *has simple obstructions.*

Proof. Suppose T has obstructions; we must find simple obstructions. So, suppose for some Δ_1, and n, and for every finite $\Delta_2 \supseteq F(\Delta_1, 3n)$, there is a finite Δ_3 and a tuple $(M^{\Delta_2}, \sigma^{\Delta_2}, A^{\Delta_2}, k^{\Delta_2})$ such that: $(M^{\Delta_2}, \sigma^{\Delta_2}) \models T_\sigma$, A^{Δ_2} is a finite subset of M^{Δ_2} and $\mathbf{b}^{\Delta_2}, \sigma(\mathbf{b}^{\Delta_2}), \mathbf{c}^{\Delta_2}, \mathbf{a}^{\Delta_2}$ contained in M^{Δ_2} are a $(\sigma^{\Delta_2}, \Delta_1, \Delta_2, \Delta_3, n)$ obstruction of length k^{Δ_2} over A^{Δ_2}. Without loss of generality $\lg(\mathbf{a}) = m = m(\Delta_1, 3n)$ and we can write $\Delta_3 = \Delta_3(\Delta_2)$. Now, define a family of simple obstruction by replacing each component of the given sequence of obstructions by an appropriate object with left prefix sim.

$$^{\text{sim}}A^{\Delta_2} = A^{\Delta_2} \cup \{\mathbf{b}^{\Delta_2}, \sigma(\mathbf{b}^{\Delta_2}), \mathbf{c}^{\Delta_2}\}$$

$$^{\text{sim}}(\mathbf{b}^{\Delta_2}) =^{\text{sim}} (\mathbf{b}'^{\Delta_2}) = \sigma(\mathbf{b}^{\Delta_2})$$

$$^{\text{sim}}\mathbf{c}^{\Delta_2} = \mathbf{c}^{\Delta_2}.$$

We use the same sequence of formulas for the Δ_2 and $\Delta_3(\Delta_2)$. It is routine to check that we now have an obstruction with respect to the identity.

Lemma 3.8 *If* T *is a stable theory with the finite cover property then* T *has obstructions.*

Proof. By II.4.1.14 of [8], there is a formula $E(\mathbf{x}, \mathbf{y}, \mathbf{z})$ such that for each $\mathbf{d}$, $E(\mathbf{x}, \mathbf{y}, \mathbf{z})$ is an equivalence relation and for each n there is a $\mathbf{d}_n$ such that $E(\mathbf{x}, \mathbf{y}, \mathbf{d}_n)$ has finitely many classes but more than n. Let Δ_1

be $\{E(\mathbf{x},\mathbf{y},\mathbf{z}), \neg E(\mathbf{x},\mathbf{y},\mathbf{z})\}$ and consider any Δ_2. Fix $\lg(\mathbf{x}) = \lg(\mathbf{y}) = r$. There are arbitrarily large sequences $\mathbf{b}_n = \langle \mathbf{b}^n_j : j < n\rangle, \mathbf{c}_n = \langle \mathbf{c}^n_j : j < n\rangle$ such that for some $\mathbf{d}_{n_1}, \mathbf{d}_{n_2}$, $\mathbf{b}_n$ is a set of representatives for distinct classes of $E(\mathbf{x},\mathbf{y},\mathbf{d}^n_1)$ while $\mathbf{c}_n$ is a set of representatives of distinct classes for $E(\mathbf{x},\mathbf{y},\mathbf{d}^n_2)$ and $n_1 < n_2$. So by compactness and Ramsey, for any Δ_2 we can find such $\mathbf{b}_k, \mathbf{c}_k$ where $k = f(\Delta_2, 2r)$ and $\mathbf{b}_k, \mathbf{c}_k$ are a sequence of length k of Δ_2 indiscernibles. Now, if Δ_3 contains formulas which express that the number of equivalence classes of $E(\mathbf{x},\mathbf{y},\mathbf{z})$ is greater than n_1 and A contains $\mathbf{d}_{n_1}, \mathbf{d}_{n_2}$, we have an $(\text{identity}, \Delta_1, \Delta_2, \Delta_3, r)$ obstruction over A. (Let $\mathbf{b} = \mathbf{b}' = \mathbf{b}_k$, $\mathbf{c} = \mathbf{c}_k$.)

4 Model Companions of T_σ

In this section we establish necessary and sufficient conditions on stable T for T_σ to have a model companion. First, we notice when the model companion, if it exists, is complete.

Note that $\mathrm{acl}(\emptyset) = \mathrm{dcl}(\emptyset)$ in C^{eq}_T means every finite equivalence relation $E(\mathbf{x},\mathbf{y})$ of T is defined by a finite conjunction: $\bigwedge_{i<n} \phi_i(\mathbf{x}) \leftrightarrow \phi_i(\mathbf{y})$.

Fact 4.1 *1. If T is stable, T_σ has the amalgamation property.*

2. If, in addition, $\mathrm{acl}(\emptyset) = \mathrm{dcl}(\emptyset)$ in C^{eq}_T then T_σ has the joint embedding property.

Proof. The first part of this Lemma was proved by (Theorem 3.3 of [6]) using the definability of types. For the second part, the hypothesis implies that types over the empty set are stationary and the result follows by similar arguments.

Lemma 4.2 *Suppose T is stable and T_σ has a model companion T^*_σ.*

1. *Then T^*_σ is complete if and only if $\mathrm{acl}(\emptyset) = \mathrm{dcl}(\emptyset)$ in C^{eq}_T.*

2. *If $(M,\sigma) \models T_\sigma$ then the union of the complete diagram of M (in L) with the diagram of (M,σ) and T^*_σ is complete.*

Proof. 1) We have just seen that if $\mathrm{acl}(\emptyset) = \mathrm{dcl}(\emptyset)$ in C^{eq}_T, then T_σ has the joint embedding property; this implies in general that the model companion is complete. If $\mathrm{acl}(\emptyset) \neq \mathrm{dcl}(\emptyset)$ in C_T^{eq}, let $E(\mathbf{x}, \mathbf{y})$ be a finite equivalence relation witnessing $\mathrm{acl}(\emptyset) \neq \mathrm{dcl}(\emptyset)$. Because E is a finite equivalence relation,

$$T_1 = T_\sigma \cup \{(\forall \mathbf{x}) E(\mathbf{x}, \sigma(\mathbf{x}))\}$$

is a consistent extension of T_σ. But since

$$T_\sigma \cup \{\neg E(\mathbf{x}, \mathbf{y})\} \cup \{\phi(\mathbf{x}) \leftrightarrow \phi(\mathbf{y}) \colon \phi \in L(T)\}$$

is consistent so is

$$T_2 = T_\sigma \cup \{(\exists \mathbf{x}) \neg E(\mathbf{x}, \sigma(\mathbf{x}))\}.$$

But T_1 and T_2 are contradictory, so T_σ^* is not complete.

2) Since we have joint embedding (from amalgamation over any model) the result follows as in Fact 4.1.

We now prove the equivalence of three conditions: the first is a condition on a pair of models. The second is given by an infinite set of L_σ sentences (take the union over all finite Δ_2) and the average requires names for all elements of M. The third is expressed by a single first order sentence in L_σ. The equivalence of the first and third suffices (Theorem 4.7) to show the existence of a model companion. In fact 1 implies 2 implies 3 requires only stability; the nfcp is used to prove 3 implies 1.

Lemma 4.3 *Suppose T is stable without the fcp. Let $(M, \sigma) \models T_\sigma$, $\mathbf{a} \in M$ and suppose that (M, σ) has no σ-obstructions. Fix $\psi(\mathbf{x}, \mathbf{y}, \mathbf{z})$ with $\mathrm{lg}(\mathbf{x}) = \mathrm{lg}(\mathbf{y}) = n$ and $\mathrm{lg}(\mathbf{z}) = \mathrm{lg}(\mathbf{a}) = m$.*

The following three assertions are equivalent:

1. *There exists (N, σ), $(M, \sigma) \subseteq (N, \sigma) \models T_\sigma$ and*

$$N \models (\exists \mathbf{x}\mathbf{y})[\psi(\mathbf{x}, \mathbf{y}, \mathbf{a}) \wedge \sigma(\mathbf{x}) = \mathbf{y}].$$

2. *Fix $\Delta_1 = \{\psi(\mathbf{x}, \mathbf{y}, \mathbf{z})\}$ and without loss of generality $\mathrm{lg}(\mathbf{z}) \leq m(\Delta_1, n)$. For $k \geq 5 \cdot f(\Delta_1, 3n)$ any finite $\Delta_2 \supseteq F(\Delta_1, 3n)$ (Fact 2.1), there are $\mathbf{b}_i, \sigma(\mathbf{b}_i)\mathbf{c}_i \in {}^{3n}M$ for $i < k$ such that*

   (a) *$\langle \mathbf{b}_i \sigma(\mathbf{b}_i) \mathbf{c}_i : i < k \rangle$ is $F(\Delta_2, 3n)$-indiscernible over $\mathbf{a}$,*

   (b) *for each $i < k$, $\psi(\mathbf{b}_i, \mathbf{c}_i, \mathbf{a})$ holds,*

   (c) *For every $\mathbf{d} \in {}^{m}M$ and $\phi(\mathbf{u}, \mathbf{v}) \in \Delta_2$ we have*

$$|\{i < k : \phi(\sigma(\mathbf{b}_i), \mathbf{d}) \leftrightarrow \phi(\mathbf{c}_i, \mathbf{d})\}| \geq f(\Delta_2, 3n)/2.$$

3. *Let* $\Delta_2 = G(\Delta_1, n)$. *Then are there are* $\mathbf{b}_i, \sigma(\mathbf{b}_i)\mathbf{c}_i \in {}^{3n}M$ *for* $i < k = 5 \cdot f(\Delta_2, 3n)$ *such that :*

   (a) $\langle \mathbf{b}_i\sigma(\mathbf{b}_i)\mathbf{c}_i : i < k\rangle$ *is* $F(G(\Delta_1, n), 3n)$*-indiscernible over* $\mathbf{a}$*,*

   (b) *for each* $i < k$*,* $\psi(\mathbf{b}_i, \mathbf{c}_i, \mathbf{a})$*,*

   (c) $\lambda_{\Delta_2}(\langle\sigma(\mathbf{b}_i) : i < k\rangle, \langle\mathbf{c}_i : i < k\rangle, \mathbf{a})$.

Proof. First we show 1) implies 2). Fix $\mathbf{b}, \mathbf{c} \in N$ with $N \models \psi(\mathbf{b}, \mathbf{c}, \mathbf{a}) \wedge \sigma(\mathbf{b}) = \mathbf{c}$. For Δ_2, let $\Delta_2^+ = F(F(\Delta_2, 3n), 3n)$. For each Δ_2, choose a finite $p \subseteq \mathrm{tp_{L(T)}}(\mathbf{b}, \mathbf{c}/\mathrm{M})$ with the same $(\Delta_2^+, 2)$ rank as $\mathrm{tp}_{\Delta_2}(\mathbf{b}, \mathbf{c}/\mathrm{M})$ (so $\mathrm{tp}_{\Delta_2}(\mathbf{b}, \mathbf{c}/\mathrm{M})$ is definable over dom p). Now inductively construct an $F(\Delta_2, 3n)$-indiscernible sequence (by Fact 2.1 1)) $\langle \mathbf{b}_i, \mathbf{c}_i : i < \omega\rangle$ by choosing $\mathbf{b}_i, \mathbf{c}_i$ in M realizing the restriction of $\mathrm{tp}_{\Delta_2^+}(\mathbf{b}, \mathbf{c}/\mathrm{M})$ to dom p along with the points already chosen. Let $\mathbf{b}'_i = \sigma(\mathbf{b}_i)$. By Ramsey's Theorem for some infinite $U \subseteq \omega$, $\langle \mathbf{b}_i, \mathbf{b}'_i, \mathbf{c}_i : i \in U\rangle$ is $F(\Delta_2, 3n)$-indiscernible over $\mathbf{a}$; renumbering let $U = \omega$. Now conditions a) and b) of assertion 2) are clear. For clause c),

$$\mathrm{avg}_{\Delta_2}(\langle \mathbf{c}_\mathrm{i} : \mathrm{i} < \omega\rangle/\mathrm{M}) = \mathrm{tp}_{\Delta_2}(\mathbf{c}, \mathrm{M})$$

$$= \sigma(\mathrm{tp}_{\Delta_2}(\mathbf{b}, \mathrm{M})) = \sigma(\mathrm{avg}_{\Delta_2}(\langle \mathbf{b}_\mathrm{i} : \mathrm{i} < \omega\rangle/\mathrm{M}))$$

since $\sigma(\mathbf{b}) = \mathbf{c}$. So, for each $\phi \in \Delta_2$ and each $\mathbf{d} \in M$ of appropriate length,

$$\phi(\mathbf{x}, \mathbf{d}) \in \mathrm{avg}_{\Delta_2}(\langle \mathbf{c}_\mathrm{i} : \mathrm{i} < \omega\rangle/\mathrm{M})$$

if and only if

$$\phi(\mathbf{x}, \sigma^{-1}(\mathbf{d})) \in \mathrm{avg}_{\Delta_2}(\langle \mathbf{b}_\mathrm{i} : \mathrm{i} < \omega\rangle/\mathrm{M}).$$

So for some $S_1, S_2 \subset \omega$ with $|S_1|, |S_2| < f(\Delta_2, 3n)/2$, we have for all $i \in \omega - (S_1 \cup S_2)$, $\phi(\mathbf{c}_i, \mathbf{d})$ if and only if $\phi(\mathbf{b}_i, \sigma^{-1}(\mathbf{d}))$. Since σ is an automorphism of M this implies for $i \in \omega - (S_1 \cup S_2)$, $\phi(\mathbf{c}_i, \mathbf{d})$ if and only if $\phi(\sigma(\mathbf{b}_i), \mathbf{d})$ which gives condition c) by using the first k elements of $\langle \mathbf{b}_i, \sigma(\mathbf{b}_i)\mathbf{c}_i : i < \omega - S_1 \cup S_2\rangle$.

3) is a special case of 2). To see this, note that 3c) is easily implied by the form analogous to 2c): For every $m \leq m(\Delta_1, n)$ and $\mathbf{d} \in {}^mM$ and $\phi(\mathbf{u}, \mathbf{v}) \in G(\Delta_1, n)$ we have

$$|\{i < k : \phi(\sigma(\mathbf{b}_i), \mathbf{d}) \leftrightarrow \phi(\mathbf{c}_i, \mathbf{d})\}| \geq f(\Delta_2, 3n)/2.$$

If T does not have f.c.p. the converse holds and we use that fact implicitly in the following argument. It remains only to show that 3) implies 1) with

$\Delta_1 = \{\psi\}$ and $\Delta_2 = G(\Delta_1, n)$. Without loss of generality we may assume N is $\aleph_1$-saturated. We claim the type

$$\Gamma = \{\psi(\mathbf{x}, \mathbf{y}, \mathbf{a})\} \cup \{\phi(\mathbf{x}, \mathbf{d}) \leftrightarrow \phi(\mathbf{y}, \sigma(\mathbf{d})) : \mathbf{d} \in M, \phi \in L(T)\} \cup \text{diag } (M)$$

is consistent. This clearly suffices.

Let $k = f(\Delta_2, 3n)$. Suppose $\langle \mathbf{b}_i \sigma(\mathbf{b}_i)\mathbf{c}_i : i < k\rangle \mathbf{a}$ satisfy 3). Let Γ_0 be a finite subset of Γ and suppose only formulas from the finite set Δ_3 and only parameters from the finite set A appear in Γ_0. Write $\mathbf{b}'_i$ for $\sigma(\mathbf{b}_i)$.

Now $\langle \mathbf{b}_i \mathbf{b}'_i \mathbf{c}_i : i \leq f(\Delta_2, 3n)\rangle \mathbf{a}$ easily satisfy the first two conditions of Definition 3.3 for being a $(\Delta_1, \Delta_2, \Delta_3, n)$-obstruction over A and, in view of Fact 2.2 3), 4), the third is given by condition 3c). Since there is no obstruction the 4th condition must fail. So there exist $\mathbf{b}^*, (\mathbf{b}^*)', \mathbf{c}^*$ so that

$$M \models \tau_{\Delta_1}(\mathbf{b}^*, (\mathbf{b}*)', \mathbf{c}^*, \langle \mathbf{b}_i, \mathbf{b}'_i, \mathbf{c}_i : i < k\rangle \mathbf{a}).$$

and $\text{tp}_{\Delta_3}((\mathbf{b}^*)'/\text{A}) = \text{tp}_{\Delta_3}(\mathbf{c}^*/\text{A})$ so Γ_0 is satisfiable.

As we'll note in Theorem 4.7, we have established a sufficient condition for T_σ to have a model companion. The next argument shows it is also necessary.

Lemma 4.4 *Suppose T is stable; if T has an obstruction then T_σ does not have a model companion.*

More precisely, suppose for some Δ_1, and n, and for every finite $\Delta_2 \supseteq F(\Delta_1, 3n)$, there is a a finite Δ_3 and a tuple $(M^{\Delta_2}, \sigma^{\Delta_2}, A^{\Delta_2}, k^{\Delta_2})$ such that: $(M^{\Delta_2}, \sigma^{\Delta_2}) \models T_\sigma$, A^{Δ_2} is a finite subset of M^{Δ_2}, $\mathbf{b}^{\Delta_2}, \sigma(\mathbf{b}^{\Delta_2}), \mathbf{c}^{\Delta_2}, \mathbf{a}^{\Delta_2}$ contained in M^{Δ_2} are a $(\sigma^{\Delta_2}, \Delta_1, \Delta_2, \Delta_3, n)$ obstruction of length k^{Δ_2} over A^{Δ_2}. Without loss of generality $\text{lg}(\mathbf{a}) = m = m(\Delta_1, 3n)$ *and we can write $\Delta_3 = \Delta_3(\Delta_2)$.*

Then the collection $\mathbf{K}_\sigma$ of existentially closed models of T_σ is not an elementary class.

Proof. We may assume T does not have f.c.p., since if it does we know by Winkler and Kudaibergerov that T_σ does not have a model companion. By the usual coding we may assume $\Delta_1 = \{\psi(\mathbf{x}, \mathbf{y}, \mathbf{z})\}$ with $\text{lg}(\mathbf{x}) = \text{lg}(\mathbf{y}) = n$, $\text{lg}(\mathbf{z}) = m$, $k = f(\Delta_1, 3n)$. Without loss of generality each M^{Δ_2} is existentially closed. Let $\mathcal{D}$ be a nonprincipal ultrafilter on $Y = \{\Delta_2 : F(\Delta_1, 3n) \subseteq \Delta_2 \subset_\omega L(T)\}$ such that for any $\Delta \in Y$ the family of supersets in Y of Δ is in $\mathcal{D}$. Expand the language to L^+ by adding a unary function symbol

σ and a $3n$-ary relation symbol Q and constants $\mathbf{a}$. Expand each of the M^{Δ_2} to an L^+-structure N_{Δ_2} by interpreting $\mathbf{a}$ as $\mathbf{a}^{\Delta_2}$ σ as σ^{Δ_2} and Q as the set $\{\mathbf{b}_i^{\Delta_2}, \sigma(\mathbf{b}_i^{\Delta_2})\mathbf{c}_i^{\Delta_2} : i < k^{\Delta_2}\}$ of $3n$-tuples. Let $(N^*, \sigma^*, Q^*, \mathbf{a}^*)$ be the ultraproduct of the N_{Δ_2} modulo $\mathcal{D}$. Let A denote $P^*(N^*)$, $\mathbf{a}^*$ denote the ultraproduct of the $\mathbf{a}^{\Delta_2}$, and $\langle \mathbf{b}_i, \mathbf{b}_i'\mathbf{c}_i : i \in I\rangle$ enumerate $Q^*(N^*)$. Now we claim

Claim 4.5 *1.* $\lg(\mathbf{b}_i) = \lg(\sigma(\mathbf{b}_i)) = \lg(\mathbf{c}) = n$ *;* $\lg(\mathbf{a}) = m$ *and* $\sigma^*(\mathbf{b}_i) = \mathbf{b}_i'$.

2. $\langle \mathbf{b}_i\mathbf{b}_i'\mathbf{c}_i : i \in I\rangle$ *is a sequence of* $L(T)$*-indiscernibles over* $\mathbf{a}^*$.

3. For each finite $\Delta_2 \subseteq L(T)$ *with* $\Delta_2 \subseteq F(\Delta_1, 3n)$ *and each finite subsequence from* $\langle \mathbf{b}_i\mathbf{b}_i'\mathbf{c}_i : i \in I\rangle$ *indexed by* J *of length at least* $k = f(\Delta_2, 3n)$ *the* Δ_2*-type of* $\langle \mathbf{b}_i\mathbf{b}_i'\mathbf{c}_i : i \in J\rangle\mathbf{a}$ *is the* Δ_2*-type of some* $(\sigma^{\Delta_2}, \Delta_1, \Delta_2, \Delta_3, n)$*-obstruction* $\langle \mathbf{b}^{\Delta_2}, \sigma(\mathbf{b}^{\Delta_2})\mathbf{c}^{\Delta_2}\rangle\mathbf{a}^{\Delta_2}$ *in* M^{Δ_2} *over the empty set.*

4. $(\mathrm{avg_L}(\langle \mathbf{b}_\mathrm{i}' : \mathrm{i} \in \mathrm{I}\rangle/\mathrm{N}) = \mathrm{avg_L}(\langle \mathbf{c}_\mathrm{i} : \mathrm{i} \in \mathrm{I}\rangle/\mathrm{N})$.

This claim follows directly from the properties of ultraproducts. (For item 3, apply Fact 3.4 and the definition of the ultrafilter D.)

Let Γ be the L-type in the variables $\langle \mathbf{x}_i\mathbf{x}_i'\mathbf{y}_i : i \in I\rangle \cup \{\mathbf{z}\}$ over the empty set of $\langle \mathbf{b}_i, \mathbf{b}_i'\mathbf{c}_i : i \in I\rangle\mathbf{a}^*$. For any finite $\Delta \subset L(T)$, let $\chi_{\Delta,k}(\overline{\mathbf{x}}, \overline{\mathbf{x}}'\overline{\mathbf{y}}, \mathbf{z})$ be the Δ-type over the empty set of a subsequence of k elements from $\langle \mathbf{b}_i\mathbf{b}_i'\mathbf{c}_i : i \in I\rangle$ and $\mathbf{a}^*$ from a realization of Γ with $\mathbf{z}$ for $\mathbf{a}^*$.

Recall the definition of τ_{Δ_1} from Notation 3.1. Let $r = f(\Delta_1, n)$ and let $\theta_{\Delta_1}(\mathbf{x}_0, \ldots \mathbf{x}_{r-1}, \mathbf{x}_0', \ldots \mathbf{x}_{r-1}', \mathbf{y}_0, \ldots \mathbf{y}_{r-1}, \mathbf{z})$ be the formula:
$(\exists \mathbf{x}, \mathbf{x}', \mathbf{y})[\chi_{\Delta_1,r}(\mathbf{x}_0, \ldots \mathbf{x}_{r-1}, \mathbf{x}_0', \ldots \mathbf{x}_{r-1}', \mathbf{y}_0, \ldots \mathbf{y}_{r-1}, \mathbf{z})$
$\wedge \tau_{\Delta_1}(\mathbf{x}, \mathbf{x}', \mathbf{y}, \mathbf{x}_0, \ldots \mathbf{x}_{r-1}, \mathbf{x}_0', \ldots \mathbf{x}_{r-1}', \mathbf{y}_0, \ldots \mathbf{y}_{r-1}, \mathbf{z}) \wedge \sigma(\mathbf{x}') = \mathbf{y}]$.

Without loss of generality we assume $0, 1 \ldots r-1$ index disjoint sequences. Now we claim:

Claim 4.6 *If* $\mathbf{K}_\sigma$*, the family of existentially closed models of* T_σ*, is axiomatized by* T_σ^**,*

$$T_\sigma^* \cup \Gamma \cup \{\sigma(\mathbf{x}_i) = \mathbf{x}_i' : i \in I\} \vdash \theta_{\Delta_1}(\mathbf{x}_0, \ldots \mathbf{x}_{r-1}, \mathbf{x}_0', \ldots \mathbf{x}_{r-1}', \mathbf{y}_0, \ldots \mathbf{y}_{r-1}, \mathbf{z}).$$

(Abusing notation we write this with the $\mathbf{x}_i, \mathbf{x}_i', \mathbf{y}_i$ free.)

For this, let $(M', \sigma') \models T^*_\sigma$ such that $\langle \mathbf{b}, \mathbf{b}'_i, \mathbf{c}_i : i \in I\rangle \mathbf{a}$ satisfy Γ in M' and for each $i \in I$, $\sigma(\mathbf{b}_i) = \mathbf{b}'_i$. Suppose $M' \mid L \prec M''$ and M'' is an $|M'|^+$-saturated model of T. In M'' we can find $\mathbf{b}\mathbf{b}', \mathbf{c}$ realizing the average of $\langle \mathbf{b}_i \mathbf{b}'_i \mathbf{c}_i : i \in I\rangle$ over M'. Then

$$\sigma'(\mathrm{tp}(\mathbf{b}/\mathrm{M}')) = \sigma'(\mathrm{avg}(\langle \mathbf{b}_\mathrm{i} : \mathrm{i} \in \mathrm{I}\rangle/\mathrm{M}')) = \mathrm{avg}(\langle \mathbf{b}'_\mathrm{i} : \mathrm{i} \in \mathrm{I}\rangle/\mathrm{M}')$$

$$= \mathrm{avg}(\langle \mathbf{c}_\mathrm{i} : \mathrm{i} \in \mathrm{I}\rangle/\mathrm{M}') = (\mathrm{tp}(\mathbf{c}/\mathrm{M}')$$

(The first and last equalities are by the choice of $\mathbf{b}, \mathbf{b}', \mathbf{c}$; the second holds since for each i, $\sigma'(\mathbf{b}_i) = \mathbf{b}'_i$, the third follows from clause 4 in the description of the ultraproduct.) Now since M'' is $|M'|^+$-saturated there is an automorphism σ'' of N extending σ' and taking $\mathbf{b}$ to $\mathbf{c}$.

As $(M', \sigma') \models T^*_\sigma$, it is existentially closed. So we can pull $\mathbf{b}, \mathbf{b}'\mathbf{c}$ down to M'. Thus, $(M', \sigma') \models \theta_{\Delta_1}(\mathbf{b}_0, \ldots \mathbf{b}_{r-1}, \mathbf{b}'_0, \ldots \mathbf{b}'_{r-1}, \mathbf{c}_0, \ldots \mathbf{c}_{r-1}, \mathbf{a})$. But (M', σ') was an arbitrary model of $T^*_\sigma \cup \Gamma \cup \{\sigma(\mathbf{x}_i) = \mathbf{x}'_i : i \in I\}$; so

$$T^*_\sigma \cup \Gamma \cup \{\sigma(\mathbf{x}_i) = \mathbf{x}'_i : i \in I\} \vdash$$
$$\theta_{\Delta_1}(\mathbf{x}_0, \ldots \mathbf{x}_{r-1}, \mathbf{x}'_0, \ldots \mathbf{x}'_{r-1}, \mathbf{y}_0, \ldots \mathbf{y}_{r-1}, \mathbf{z}).$$

By compactness, some finite subset Γ_0 of Γ and a finite number of the specifications of σ suffice; let Δ^* be the formulas mentioned in Γ_0 along with those in $F(\Delta_1, 3n)$ and k the number of x_i, y_i appearing in Γ_0 and let $\Delta_2 = F(\Delta^*, n)$. Without loss of generality, $k \geq f(\Delta_1, 3n)$. Then, $T^*_\sigma \vdash$

$$(\forall \mathbf{x}_0 \ldots \mathbf{x}_{k-1}, \mathbf{x}'_0, \ldots \mathbf{x}'_{k-1}, \mathbf{y}_0, \ldots \mathbf{y}_{k-1})[(\chi_{\Delta_2,k}(\mathbf{x}_0, \ldots \mathbf{x}_{k-1}, \mathbf{x}'_0, \ldots \mathbf{x}'_{k-1}, \mathbf{y}_0, \ldots \mathbf{y}_{k-1}, \mathbf{z})$$

$$\wedge \bigwedge_{i<k} \sigma(\mathbf{x}_i) = \mathbf{x}'_i) \rightarrow \theta_{\Delta_1}(\mathbf{x}_0, \ldots \mathbf{x}_{r-1}, \mathbf{x}'_0, \ldots \mathbf{x}'_{r-1}, \mathbf{y}_0, \ldots \mathbf{y}_{r-1}, \mathbf{z})].$$

By item 3) in Claim 4.5, fix a Δ_2 and $\Delta_3 = \Delta_3(\Delta_2, m)$ containing Δ_2 and $\langle \mathbf{b}_i^{\Delta_2} \mathbf{b'}_i^{\Delta_2} \mathbf{c}_i^{\Delta_2} : i < k\rangle \mathbf{a}^{\Delta_2}$ which form a $(\sigma, \Delta_1, \Delta_2, \Delta_3, n)$-obstruction over $\mathbf{a}^{\Delta_2}$ and so that:

$$M^{\Delta_2} \models \chi_{\Delta_2,k}(\langle \mathbf{b}_i^{\Delta_2} \mathbf{b'}_i^{\Delta_2} \mathbf{c}_i^{\Delta_2} : i < k\rangle \mathbf{a}^{\Delta_2}).$$

By the definition of an obstruction, $\sigma(\mathbf{b}_i^{\Delta_2}) = \mathbf{b'}_i^{\Delta_2}$. So by the choice of Γ_0,

$$(M^{\Delta_2}, \sigma^{\Delta_2}) \models \theta_{\Delta_1}(\langle \mathbf{b}_i^{\Delta_2} \mathbf{b'}_i^{\Delta_2} \mathbf{c}_i^{\Delta_2} : i < r\rangle \mathbf{a}^{\Delta_2}).$$

Now, let $\mathbf{b}, \mathbf{b}'\mathbf{c} \in M^{\Delta_2}$ with $\sigma^{\Delta_2}(\mathbf{b}') = \mathbf{c}$ witness this sentence. Then

$$(M^{\Delta_2}, \sigma^{\Delta_2}) \models \tau_{\Delta_1}(\langle \mathbf{b}_i^{\Delta_2} \mathbf{b'}_i^{\Delta_2} \mathbf{c}_i^{\Delta_2} : i < k\rangle \mathbf{a}^{\Delta_2})$$

By the definition of obstruction,

$$\sigma^{\Delta_2}(\mathrm{tp}_{\Delta_3}(\mathbf{b}'/\mathrm{A}^{\Delta_2}\cup\bigcup_{i<k}\langle\mathbf{b}_i^{\Delta_2}\sigma(\mathbf{b}_i^{\Delta_2})\mathbf{c}_i\rangle\cup\mathbf{a}^{\Delta_2}))\neq\mathrm{tp}_{\Delta_3}(\mathbf{c}/\mathrm{A}^{\Delta_2}\cup\bigcup_{i<k}\langle\mathbf{b}_i^{\Delta_2}\sigma(\mathbf{b}_i^{\Delta_2})\mathbf{c}_i^{\Delta_2}\rangle\cup\mathbf{a}^{\Delta_2}).$$

This contradicts that σ^{Δ_2} is an automorphism and we finish.

Finally we have the main result.

Theorem 4.7 *If T is a stable theory, T_σ has a model companion if and only if T_σ admits no σ-obstructions.*

Proof. We showed in Lemma 4.4 that if T_σ has a model companion then there is no obstruction. If there is no obstruction, Lemma 3.8 implies T does not have the finite cover property. By Lemma 4.3 for every formula $\psi(\mathbf{x},\mathbf{y},\mathbf{z})$ there is an L-formula $\theta_\psi(\mathbf{z})$ (write out condition 3 of Lemma 4.3) which for any $(M,\sigma)\models T_\sigma$ holds of any $\mathbf{a}$ in M if and only if there exists (N,σ), $(M,\sigma)\subseteq(N,\sigma)\models T_\sigma$ and

$$(N,\sigma)\models(\exists\mathbf{x}\mathbf{y})\psi(\mathbf{x},\mathbf{y},\mathbf{a})\wedge\sigma(\mathbf{x})=\mathbf{y}.$$

Thus, the class of existentially closed models of T_σ is axiomatized by the sentences: $(\forall\mathbf{z})\theta_\psi(\mathbf{z})\rightarrow(\exists\mathbf{x}\mathbf{y})\psi(\mathbf{x},\mathbf{y},\mathbf{a})\wedge\sigma(\mathbf{x})=\mathbf{y}$. (We can restrict to formulas of the form $\psi(\mathbf{x},\sigma(\mathbf{x}),\mathbf{a})$ by the standard trick ([4, 1]).

Kikyo and Pillay [4] note that if a strongly minimal theory has the definable mulplicity property then T_σ has a model companion. In view of Theorem 4.7, this implies that if T has the definable multiplicity property, then T_σ admits no σ-obstructions. Kikyo and Pillay conjecture that for a strongly minimal set, the converse holds: if T_σ has a model companion then T has the definable multiplicity property. They prove this result if T is a finite cover of a theory with the finite multiplicity property. It would follow in general from a positive answer to the following question.

Question 4.8 *If the strongly minimal theory T with finite rank does not have the definable multiplicity property, must it omit obstructions?*

Pillay has given a direct proof that if a strongly minimal T has the definable multiplicity property, then T_σ admits no σ-obstructions. Pillay has provided an insightful reworking of the ideas here in a note which is available on his website [7].

Here is a final question:

Question 4.9 *Can T_σ for an $\aleph_0$-categorical T admit obstructions?*

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