TERMINATING PROOF SEARCH IN ELEMENTARY GEOMETRY

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Terminating proof search in elementary geometry

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Abstract: Plane projective and affine geometries are formalized as systems of mathematical rules added to the contraction-free sequent calculus *G3c*. By the general result of Negri and von Plato (1998) for such extensions, all structural rules are eliminable. It is shown that in a minimum-size derivation, all terms in the derivation are maximal terms in the endsequent. Decidability of the geometries for quantifier-free sequents follows through termination of root-first proof search. As an example, the independence of Euclid's fifth postulate from the other affine axioms is shown.

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1. Introduction

In a previous work Negri and von Plato (1998), a general method was given for converting mathematical axioms into rules of proof. When such rules are added to suitable sequent calculi, the analysis of the structure of derivations in sequent calculi for pure logic carries over to the mathematical theories. We shall here apply the method to plane projective and affine geometry. The sequent calculus we use is what is called calculus *G3c*, a classical calculus with the remarkable properties that the logical rules for the connectives are invertible and the structural rules eliminable from derivations. Moreover, the elimination of weakening and contraction is *height preserving*, so that these rules may be applied at any stage of a derivation without affecting its size.

Proof search in the calculus *G3c* proceeds root first, from the endsequent to be derived. Whenever a rule is instantiated, its conclusion uniquely determines the premisses. For the propositional part of *G3c*, each root-first step decomposes a formula until no rule can be instantiated. By the invertibility of the propositional rules of *G3c*, the derivability of a quantifier-free sequent $\Gamma \rightarrow \Delta$ is equivalent to the derivability of a finite number of sequents of the form

$$P_1, \ldots, P_m \rightarrow Q_1, \ldots, Q_n$$

with atoms P_i, Q_j. Following Gentzen, we call sequents of this form *basic mathematical sequents*. By the subformula property, only mathematical rules are used in cut-free derivations of such sequents. Thus, the task of proof search is divided into a logical and a mathematical part. For the proof analysis of specific theories, only the latter part need be considered.

In mathematical rules, the subformula property of cut-free derivations is somewhat weaker than the standard property for purely logical calculi: The rules remove atomic formulas from premisses, and it can also happen that these atoms contain terms that are not determined from the conclusion. We show that the systems of rules of projective and affine geometry have instead a *subterm* property for minimum-size derivations: All terms in such derivations are terms of the endsequent. Moreover, all maximal terms are maximal terms in the endsequent, not hidden as parts in other terms. By the subterm property, only a bounded number of different atoms are instantiated in premisses of rules when applied in a root-first order, and the admissibility of height-preserving contraction excludes duplications. Termination of proof search and decidability of the geometries for quantifier-free sequents follow. The height of derivation of basic mathematical sequents is bounded by the square of the number of maximal terms in the endsequent.

As an example of the method of proof analysis through contraction-free systems of mathematical rules, the independence of Euclid's fifth postulate from the other axioms of affine plane geometry is shown.

2. Left and right rules

Through the division of proof search into separate logical and mathematical parts, we only need to consider mathematical rules. However, for the sake of completeness we briefly present the logical calculus *G3c* also. Sequents $\Gamma \to \Delta$ have finite multisets of formulas Γ and Δ as antecedents and succedents. Atomic formulas are denoted by $P, Q, R, \ldots$ and arbitrary formulas by $A, B, C, \ldots$. We only consider the propositional rules of *G3c* here:

Initial sequents:

$$P, \Gamma \to \Delta, P$$

Logical rules:

$$\frac{A, B, \Gamma \to \Delta}{A\&B, \Gamma \to \Delta}\, L\& \qquad \frac{\Gamma \to \Delta, A \quad \Gamma \to \Delta, B}{\Gamma \to \Delta, A\&B}\, R\&$$

$$\frac{A, \Gamma \to \Delta \quad B, \Gamma \to \Delta}{A \vee B, \Gamma \to \Delta}\, L\vee \qquad \frac{\Gamma \to \Delta, A, B}{\Gamma \to \Delta, A \vee B}\, R\vee$$

$$\frac{\Gamma \to \Delta, A \quad B, \Gamma \to \Delta}{A \supset B, \Gamma \to \Delta}\, L\supset \qquad \frac{A, \Gamma \to \Delta, B}{\Gamma \to \Delta, A \supset B}\, R\supset$$

$$\frac{}{\bot, \Gamma \to \Delta}\, L\bot$$

The false formula $\bot$ is not considered atomic, but a zero-place logical operation with a zero-premiss left rule and no right rule. Negation $\sim A$ is defined as $A \supset \bot$.

In the calculus *G3c*, the structural rules of weakening and contraction are not only admissible, but this admissibility is *height preserving:* If the premiss of a weakening or contraction is derivable, its conclusion is derivable without the weakening or contraction step and with the same height of derivation as the premiss. Thus, we can add at any stage weakening formulas, and contract duplications, without affecting the size of the derivation. Height-preserving admissibility of weakening and contraction is a result of the subtle design of the calculus *G3c*.

Starting with the calculus *G3c*, the structural rules remain admissible also in extensions of *G3c* by suitably formulated mathematical rules. Given a quantifier-free formula A (axiom of some theory), it is equivalent to a conjunction of disjunctions of atoms and negations of atoms $\sim P_1 \vee \ldots \vee \sim P_m \vee Q_1 \vee \ldots \vee Q_n$. Each such conjunct can be converted into the equivalent form $P_1 \& \ldots \& P_m \supset Q_1 \vee \ldots \vee Q_n$ in which the antecedent and implication are absent if $m = 0$ and the consequent is $\perp$ if $n = 0$. It is shown in Negri and von Plato (1998) that these formulas are representable as systems of sequent calculus rules for which the structural rules are admissible. Formulas are converted into rules through one of the following general schemes:

Left and right schemes for mathematical rules:

$$\frac{Q_1, P_1, \ldots, P_m, \Gamma \rightarrow \Delta \quad \ldots \quad Q_n, P_1, \ldots, P_m, \Gamma \rightarrow \Delta}{P_1, \ldots, P_m, \Gamma \rightarrow \Delta}\textit{L-rule}$$

$$\frac{\Gamma \rightarrow \Delta, Q_1, \ldots, Q_n, P_1 \quad \ldots \quad \Gamma \rightarrow \Delta, Q_1, \ldots, Q_n, P_m}{\Gamma \rightarrow \Delta, Q_1, \ldots, Q_n}\textit{R-rule}$$

Addition of a rule following one of the schemes to *G3c* will make sequents $\rightarrow P_1 \& \ldots \& P_m \supset Q_1 \vee \ldots \vee Q_n$ derivable by the logical rules. The right scheme is more intuitive: Omitting Δ, if each of the P_i follow from assumptions Γ, the cases under Γ are $Q_1, \ldots, Q_n$. The left rule-scheme instead states that if something follows from each of the cases $Q_1, \ldots, Q_n$, then it already follows from the P_i together.

As explained in Negri and von Plato (1998), the *principal formulas* $P_1, \ldots, P_m$ for the left scheme, and $Q_1, \ldots, Q_n$ for the right one, have to be repeated in each premiss as assumptions and cases, respectively. Such repetition is needed for proving the rule of contraction admissible. We also note the following

Closure condition: *Given a system with left mathematical rules, if it has a rule with an instance of the form*

$$\frac{Q_1, P_1, \ldots, P_{m-2}, P, P, \Gamma \rightarrow \Delta \quad \ldots \quad Q_n, P_1, \ldots, P_{m-2}, P, P, \Gamma \rightarrow \Delta}{P_1, \ldots, P_{m-2}, P, P, \Gamma \rightarrow \Delta}\textit{L-rule}$$

then also the rule

$$\frac{Q_1, P_1, \ldots, P_{m-2}, P, \Gamma \rightarrow \Delta \quad \ldots \quad Q_n, P_1, \ldots, P_{m-2}, P, \Gamma \rightarrow \Delta}{P_1, \ldots, P_{m-2}, P, \Gamma \rightarrow \Delta}\textit{L-rule}$$

has to be included in the system, and similarly for the right rule-scheme.

It can happen that a substitution in the atoms of a rule produces a duplication P, P of formulas the contraction of which into P requires the condition. But it is in principle unproblematic, because the number of rules to be added to a given system of mathematical rules is bounded. Often the condition does not arise at all.

Theorem 2.1. *Height-preserving weakening and contraction and the rule of cut are admissible in extensions of G3c following the left rule-scheme and satisfying the closure condition, and similarly for the right rule-scheme.*

This result is proved for left rules in Negri and von Plato (1998, section 3). The result for the right rule-scheme has a dual proof.

We note that the subformula property fails for mathematical rules, because rules remove atoms from premisses. This restriction has, however, only a limited effect on proof analysis. We have, by the admissibility of cut, the

Corollary 2.2. *If a basic mathematical sequent* $\Gamma \rightarrow \Delta$ *is derivable, its derivation uses only mathematical rules.*

Lemma 2.3. *The logical rules of G3c commute down with mathematical rules.*

Proof: A routine verification. QED.

Logical rules are conservative over mathematical rules for basic mathematical sequents. Each quantifier-free formula is equivalent to a conjunction of formulas $P_1 \& \ldots \& P_m \supset Q_1 \vee \ldots \vee Q_n$, and by the invertibility of the logical rules of *G3c*, these are derivable if and only if the corresponding basic mathematical sequents $P_1, \ldots, P_m \rightarrow Q_1, \ldots, Q_n$ are derivable. We consider only derivations of such "logic-free" sequents in what follows.

For a detailed introduction to sequent calculus and its extension by left mathematical rules we refer to the book Negri and von Plato (2001). Dual systems with right rules first appear in Negri, von Plato, and Coquand (2001).

3. Plane projective geometry as an axiomatic system

We have two sets of basic objects, *points* $a, b, c, \ldots$ and *lines* $l, m, n, \ldots$. The basic concepts of projective geometry are as follows:

(2.1) $a = b$, a and b are *equal* points,
(2.2) $l = m$, l and m are *equal* lines,
(2.3) $a \in l$, point a is *incident* with line l.

Next we have two constructions:

(2.4) $ln(a,b)$, the *connecting line* of points a, b,
(2.5) $pt(l,m)$, the *intersection point* of lines l, m.

In (2.4), the *degeneracy* $a = b$ is possible, and in (2.5), the degeneracy $l = m$. The geometrical axioms are as follows:

I Axioms for equality relations:

$$a = a, \quad a = b \;\&\; b = c \supset a = c, \quad a = b \supset b = a,$$
$$l = l, \quad l = m \;\&\; m = n \supset l = n, \quad l = m \supset m = n.$$

Constructed objects obey incidence properties expressed by the next group of axioms, in which the degeneracies $a = b$ and $l = m$, respectively, are included as cases:

II Projective axioms of incidence:

$$a = b \vee a \in ln(a,b), \quad a = b \vee b \in ln(a,b),$$
$$l = m \vee pt(l,m) \in l, \quad l = m \vee pt(l,m) \in m.$$

Uniqueness of connecting lines and intersection points is guaranteed by an axiom found by Skolem (1920):

III Projective uniqueness axiom:

$$a \in l \;\&\; a \in m \;\&\; b \in l \;\&\; b \in m \supset a = b \vee l = m.$$

Finally, we need principles that guarantee the substitutability of equals:

IV Projective substitution axioms:

$$a \in l \;\&\; a = b \supset b \in l, \quad a \in l \;\&\; l = m \supset a \in m.$$

These axioms are variants of standard formulations, except for the axioms of group II that make explicit the degeneracies. Usually there is a further axiom stating the existence of at least three noncollinear points, but we do not need to use such existential axioms. With the listing of assumptions and cases in sequent calculus, the same effect is achieved if the sequent to be derived contains the cases $a = b$ and $c \in ln(a,b)$ in the succedent.

4. Proof analysis in projective geometry

We shall use the left rule-scheme for the geometrical rules. They are grouped as in the axiomatic presentation in Section 3.

I Rules for equality relations:

$$\frac{a = a, \Gamma \to \Delta}{\Gamma \to \Delta}\,\textit{Ref} \qquad \frac{a = c, a = b, b = c, \Gamma \to \Delta}{a = b, b = c, \Gamma \to \Delta}\,\textit{Trans} \qquad \frac{b = a, a = b, \Gamma \to \Delta}{a = b, \Gamma \to \Delta}\,\textit{Sym}$$

$$\frac{l = l, \Gamma \to \Delta}{\Gamma \to \Delta}\,\textit{Ref} \qquad \frac{l = n, l = m, m = n, \Gamma \to \Delta}{l = m, m = n, \Gamma \to \Delta}\,\textit{Trans} \qquad \frac{m = l, l = m, \Gamma \to \Delta}{l = m, \Gamma \to \Delta}\,\textit{Sym}$$

II Rules for incidence:

$$\frac{a=b, \Gamma \to \Delta \quad a \in ln(a,b), \Gamma \to \Delta}{\Gamma \to \Delta}{\scriptstyle Inc} \qquad \frac{a=b, \Gamma \to \Delta \quad b \in ln(a,b), \Gamma \to \Delta}{\Gamma \to \Delta}{\scriptstyle Inc}$$

$$\frac{l=m, \Gamma \to \Delta \quad pt(l,m) \in l, \Gamma \to \Delta}{\Gamma \to \Delta}{\scriptstyle Inc} \qquad \frac{l=m, \Gamma \to \Delta \quad pt(l,m) \in m, \Gamma \to \Delta}{\Gamma \to \Delta}{\scriptstyle Inc}$$

III Uniqueness rule:

$$\frac{a=b, a \in l, a \in m, b \in l, b \in m, \Gamma \to \Delta \quad l=m, a \in l, a \in m, b \in l, b \in m, \Gamma \to \Delta}{a \in l, a \in m, b \in l, b \in m, \Gamma \to \Delta}{\scriptstyle Uni}$$

IV Substitution rules:

$$\frac{b \in l, a \in l, a=b, \Gamma \to \Delta}{a \in l, a=b, \Gamma \to \Delta}{\scriptstyle Sub} \qquad \frac{a \in m, a \in l, l=m, \Gamma \to \Delta}{a \in l, l=m, \Gamma \to \Delta}{\scriptstyle Sub}$$

Each mathematical rule removes one atom from each premiss, so that no new constructions appear.

We notice that substitutions in the atomic formulas can produce duplications in the conclusion in the two *Trans* rules and in rule *Uni*, with $a \equiv b \equiv c$ and $l \equiv m \equiv n$ in the first two and $a \equiv b$ or $l \equiv m$ in the third. However, the contracted rule is in each case an instance of rule *Ref*, either for points or for lines. Therefore the closure condition does not lead to any additional rules. All of our rules follow the rule-scheme, and we conclude from theorem 2.1 the

Corollary 4.1. *Height-preserving weakening and contraction and the rule of cut are admissible in the sequent calculus for projective geometry.*

We shall next prove a *subterm property* for minimum-size derivations in the sequent system for projective geometry.

Definition 4.2. A *maximal* term occurrence is a term occurrence that is not a subterm of another term.

To prove the subterm property, the method is to consider a minimum-size derivation and a maximal term that disappears from the derivation through the removal of the atom or atoms in which it makes its last appearance. Next these removed atoms are traced up in the derivation until they are principal. At each such step, there will always be at least one removed atom in a premiss in which the disappeared term is maximal, or else the term is also in another principal atom. (This is just a property that our rules have.) In the first case, the removed atoms are traced up in the derivation until they are principal, and last atoms in topsequents are found, with at least one topsequent and atom in which the disappeared term is maximal. Since all topsequents are initial sequents, either this atom will make the sequent an initial sequent, with the atom containing the term also in the succedent, or it can be deleted and the derivation shortened. Deletion is possible because the atom is never active before the step that removes it. If one traced up other than removed atoms, they could not be deleted, and so the crucial point of the proof method is to *always trace the removed atoms.* In the second case, when the term is not in a removed atom, it is in a second principal atom and has to be removed further down in the derivation. At this step, it happens that there will be another principal atom to be removed further down, but this cannot go on forever.

Theorem 4.3. Subterm property. *In a minimum-size derivation of a basic mathematical sequent* $\Gamma \to \Delta$ *in plane projective geometry, all maximal terms are maximal terms in* Γ, Δ.

Proof: Consider a first point a that disappears. The rule in which it makes its last appearance is *Ref* or *Inc* on points:

$$\frac{a = a, \Gamma \to \Delta}{\Gamma \to \Delta}\,{\scriptstyle Ref} \qquad \frac{a = b, \Gamma \to \Delta \quad a \in ln(a,b), \Gamma \to \Delta}{\Gamma \to \Delta}\,{\scriptstyle Inc}$$

With $a \equiv pt(l,m)$, there is a second type of *Inc*:

$$\frac{l = m, \Gamma \to \Delta \quad pt(l,m) \in l, \Gamma \to \Delta}{\Gamma \to \Delta}\,{\scriptstyle Inc}$$

If the rule is *Ref*, $a = a$ is principal higher up in *Trans, Sym*, or *Sub*. In each case, the premiss has a duplication of an atom. By height-preserving contraction, the conclusion is derivable without the rule, contrary to the assumption of a derivation of minimum height. Therefore rule *Ref* cannot make a point disappear.

If point a disappears by rule *Inc* of the first type, we trace up the atoms $a = b$ and $a \in ln(a,b)$. For $a = b$, the rule is *Trans, Sym,* or *Sub* and the cases are

$$\frac{a = c, a = b, b = c, \Gamma' \to \Delta}{a = b, b = c, \Gamma' \to \Delta}\,{\scriptstyle Trans} \qquad \frac{c = b, c = a, a = b, \Gamma' \to \Delta}{c = a, a = b, \Gamma' \to \Delta}\,{\scriptstyle Trans}$$

$$\frac{b = a, a = b, \Gamma' \to \Delta}{a = b, \Gamma' \to \Delta}\,{\scriptstyle Sym} \qquad \frac{a \in m, b \in m, a = b, \Gamma' \to \Delta}{b \in m, a = b, \Gamma' \to \Delta}\,{\scriptstyle Sub}$$

In the first, third, and fourth cases, point a is in the removed atom and is traced further. In the second, there is another occurrence of a in the atom $c = a$. It must be removed before the step that removes the last occurrence of a, in the atom $a = b$. The rule removing $c = a$ cannot be *Ref*, for in this case we have $c \equiv a$ and there is a duplication in the premiss. Therefore the rule must be *Trans, Sym,* or *Uni.* With *Trans* or *Sym*, there will again be an equality with point a to be removed, with *Uni*, there will be an atom $a \in n$. The rules that remove it are *Inc* and *Sub*. With the latter, there will be an equality with point a to be removed. If $a \in n$ is removed by *Inc* of the first type, $n \equiv ln(a,d)$ and the other premiss has a removed equality, the degeneracy $a = d$ that is traced up.

Next consider rule *Inc* of the second type, with removed atoms $l = m$ and $pt(l,m) \in l$. Tracing up the latter, we find that in all matching rules, point $pt(l,m)$ either is in a removed atom or occurs in another principal atom.

Each rule either has point a in a removed atom of a premiss or in another principal atom that has to be removed from the conclusion. The latter leads to an infinite derivation, the former to a topsequent with an atom with point a in the succedent. Therefore no minimum-size derivation has points as maximal terms that disappear in the derivation.

The proof that no lines can disappear from a derivation of minimum size is dual to the above. QED.

Corollary 4.4. Bounded proof search and decidability. *Let $\Gamma \to \Delta$ be a basic mathematical sequent with r maximal points, s maximal lines, and t distinct atoms in Γ. Derivability of $\Gamma \to \Delta$ in plane projective geometry is decidable, with proof-search bounded by the polynomial $r^2 + s^2 + rs - t$.*

Proof: If in the instantiation of a rule a duplication in a premiss is produced, proof search fails by the admissibility of height-preserving contraction. The bound follows by considering the number of distinct atoms in r points and s lines, given that the antecedent already has t distinct atoms. QED.

Degeneracies play a crucial role in the proof of the subterm property, theorem 4.3. Without them, track is lost of the maximal terms in a derivation. In the above proof, when the equality $c = a$ was removed by *Uni*, with $a \in n$ principal, and this latter was removed by *Inc*, the term a could only be traced through the degeneracy $a = d$ in the first premiss of *Inc*.

The system with explicit degeneracies is superficially weaker than one without them. For example, an instance of *Inc* without degeneracy

$$\frac{a \in ln(a,b) \ \to \ a \in ln(a,b)}{\to \ a \in ln(a,b)}$$

derives the sequent $\to a \in ln(a,b)$ that is underivable if track is kept of degeneracies: The rules that can conclude it are *Ref* or *Inc*. The former can be the last ones applied, and there are two instantiations that give the sequent $a = a, ln(a,b) = ln(a,b) \ \to \ a \in ln(a,b)$ to be derived. Rule *Inc* instantiates a maximal term that is not maximal in the endsequent, and proof search fails.

The sequent $c \in ln(a,b) \ \to \ a \in ln(b,c)$ is likewise underivable: Rule *Ref* can instantiate four equalities in the antecedent. However, with only two maximal points, *Trans* is a useless rule, and the same for lines. Rules *Sym* and *Sub* give duplications. Instances of rule *Inc* have one of the two lines $ln(a,b)$ and $ln(b,c)$ in the right premiss, so the left premiss has either $a = b$ or $b = c$, but this is impossible because b is not a maximal term in the conclusion.

If we add the degeneracies $a=b, b=c$ to the succedent of the previous sequent, it becomes the derivable sequent $c \in ln(a,b) \rightarrow a=b, b=c, a \in ln(b,c)$. The contents of the result are more intuitive if read as a contrapositive: If points b and c are distinct, if point a is outside line $ln(b,c)$, and if points a and b are distinct, point c is outside line $ln(a,b)$. This is known as the "triangle axiom." The degeneracies have turned into *conditions* for the well-formedness of constructed lines. It turns out that line $ln(a,b)$ in the triangle axiom is well-formed also without its condition, for the sequent $a=b \rightarrow b=c, a \in ln(b,c)$ is derivable. (Consider again the contrapositive for the intuitive content.) This result is a typical "lemma" that can be combined with the main result $c \in ln(a,b) \rightarrow b=c, a \in ln(b,c), a=b$: A cut with cut formula $a=b$ gives a derivation of $c \in ln(a,b) \rightarrow b=c, a \in ln(b,c)$.

5. Proof analysis in affine geometry

To obtain an axiom system for affine geometry, the following additions and modifications are made to the projective axiomatization of Section 3:

There is one concept more:

(5.1) $\quad l \parallel m, \quad l$ and m are *parallel* lines.

The degeneracy in the intersection point construction is changed into $l \parallel m$. There is one construction more:

(5.2) $\quad par(l,a), \quad$ the *parallel* to line l through point a.

Construction (5.2) has the degeneracy $a \in l$. Equivalently, it has the traditional condition that point a be outside line l.

The additional affine axioms are:

I General axioms for parallelism:

$l \parallel l, \quad l \parallel m \,\&\, m \parallel n \supset l \parallel n, \quad l \parallel m \supset m \parallel l.$

II Affine axioms of incidence and parallelism:

$a \in l \vee a \in par(l,a), \quad a \in l \vee l \parallel par(l,a).$

III Affine uniqueness axiom:

$a \in l \,\&\, a \in m \,\&\, l \parallel m \supset l=m.$

IV Affine substitution axiom:

$l \parallel m \,\&\, m=n \supset l \parallel n.$

Degeneracies in the parallel line construction are made explicit in the axioms of group II. Standard axiomatic presentations of affine geometry, such as Artin (1957), state the existence and uniqueness of connecting lines and parallel lines, and existence and properties of intersection points are obtained through a defined notion of parallels. With a parallel line construction, it is possible to define $l \parallel m$ as $par(l,a) = par(m,a)$, but for proof analysis, a separate notion of parallelism works better.

The rules to be added to projective geometry are

I Rules for the parallelism relation:

$$\frac{l \parallel l, \Gamma \to \Delta}{\Gamma \to \Delta}{\scriptstyle Ref} \quad \frac{l \parallel n, l \parallel m, m \parallel n, \Gamma \to \Delta}{l \parallel m, m \parallel n, \Gamma \to \Delta}{\scriptstyle Trans} \quad \frac{m \parallel l, l \parallel m, \Gamma \to \Delta}{l \parallel m, \Gamma \to \Delta}{\scriptstyle Sym}$$

II Rules for incidence and parallelism:

$$\frac{a \in l, \Gamma \to \Delta \quad a \in par(l,a), \Gamma \to \Delta}{\Gamma \to \Delta}{\scriptstyle Incpar} \quad \frac{a \in l, \Gamma \to \Delta \quad l \parallel par(l,a), \Gamma \to \Delta}{\Gamma \to \Delta}{\scriptstyle Par}$$

III Uniqueness of parallels:

$$\frac{l = m, a \in l, a \in m, l \parallel m, \Gamma \to \Delta}{a \in l, a \in m, l \parallel m, \Gamma \to \Delta}{\scriptstyle Unipar}$$

IV Substitution for parallels:

$$\frac{l \parallel n, l \parallel m, m = n, \Gamma \to \Delta}{l \parallel m, m = n, \Gamma \to \Delta}{\scriptstyle Subpar}$$

Corollary 5.1. *Height-preserving weakening and contraction and the rule of cut are admissible in the sequent calculus for affine geometry.*

We shall next prove the subterm property for minimum-size derivations in the sequent system for affine geometry.

Theorem 5.2. Subterm property. *In a minimum-size derivation of a basic mathematical sequent* $\Gamma \to \Delta$ *in plane affine geometry, each maximal term is a maximal term in* Γ, Δ.

Proof: The proof is an extension of Theorem 4.3. Point a can disappear by the additional rules *Incpar* and *Par*.

For *Incpar*, tracing up the atoms $a \in l$ and $a \in par(l,a)$ the new case in which they can be principal is rule *Unipar*. The steps for each atom are:

$$\frac{l = m, a \in l, a \in m, l \parallel m, \Gamma' \to \Delta}{a \in l, a \in m, l \parallel m, \Gamma' \to \Delta}{\scriptstyle Unipar}$$

$$\frac{par(l,a) = m, a \in par(l,a), a \in m, par(l,a) \parallel m, \Gamma' \to \Delta}{a \in par(l,a), a \in m, par(l,a) \parallel m, \Gamma' \to \Delta}{\scriptstyle Unipar}$$

Point a is not maximal in removed atoms, but it is maximal in the other principal atom $a \in m$. Now the new case is that this atom is removed by *Incpar* with $a \in m \equiv a \in par(n,a)$ and there is no principal atom with a. We find instead that the other premiss of this *Incpar*, with the degeneracy, has the removed atom $a \in n$ with point a that will be traced up.

For rule *Par*, the removed atoms are $a \in l$ and $l \parallel par(l,a)$ and we trace the atom $a \in l$.

If a line l disappears, it is found similarly that l either is in a removed atom or in another principal atom, and it follows as in theorem 4.3 that there is no minimum-size derivation in which terms disappear. QED.

The subterm property depends critically on many properties that extensions of the calculus *G3c* with mathematical rules have, as well as on the specific form of the geometrical axioms. Two of the affine axioms have a seemingly redundant addition, maintained only by the archaic wisdom of school geometry, namely the degeneracy $a \in l$ in the construction $par(l,a)$. However, the proof of theorem 5.2 relies essentially on degeneracies. An example shows that the subterm property fails without these: To derive $\rightarrow l \parallel par(par(l,a),b)$ we instantiate twice with rule *Par* disregarding premisses with degeneracies:

$$\cfrac{\cfrac{l \parallel par(l,a), par(l,a) \parallel par(par(l,a),b) \rightarrow l \parallel par(par(l,a),b)}{par(l,a) \parallel par(par(l,a),b) \rightarrow l \parallel par(par(l,a),b)}}{\rightarrow l \parallel par(par(l,a),b)}$$

To derive the topsequent, we add by *Trans* the atom $l \parallel par(par(l,a),b)$ to the antecedent and obtain an initial sequent, but the middle term of transitivity $par(l,a)$ is not a maximal term of the endsequent. The construction $par(l,a)$ has the degeneracy $a \in l$ and the construction $par(par(l,a),b)$ the further degeneracy $b \in par(l,a)$. With the addition of these as cases in the succedent, the result becomes derivable in the system of rules of affine geometry with the subterm property maintained.

As an application of the system of rules of affine geometry, we consider Euclid's fifth postulate. We first derive a form of the fifth postulate from the geometrical rules: Given a point a outside line l, no point is incident with both l and the parallel to l through point a. Axiomatically, we may express this by the formula

$$\sim a \in l \supset \sim (b \in l \ \& \ b \in par(l,a))$$

It takes some effort to derive this formula from the axioms by natural deduction or some similar method of logical inference. In sequent calculus,

$$b \in l, b \in par(l,a) \rightarrow a \in l$$

expresses the same postulate as a logic-free sequent. To derive this sequent, we note that by corollary 2.2, all rules in its derivation are mathematical, and therefore the succedent is always the same, $a \in l$. Further, instantiated terms must be maximal terms in the endsequent, and no duplication of formulas is permitted. The second restriction follows from the admissibility of height-preserving contraction. The possible new terms are restricted to the two points a, b and the two lines $l, par(l, a)$. With these prescriptions, root-first proof search is very nearly deterministic. Inspecting the sequent to be derived, the last rule is *Ref*, *Incpar*, or *Par*. With two points and two lines, *Ref* will lead to duplications. Rule *Incpar* has one instance that does not produce a duplication, rule *Par* likewise, the order of the rules being irrelevant. We indicate by ... superfluous repetitions in premisses:

$$\frac{a\in l\ldots\to a\in l \qquad \dfrac{a\in l\ldots\to a\in l \quad l \,\|\, par(l,a), a\in par(l,a), b\in l, b\in par(l,a)\to a\in l}{a\in par(l,a), b\in l, b\in par(l,a)\to a\in l}{\scriptstyle Par}}{b\in l, b\in par(l,a)\to a\in l}{\scriptstyle Incpar}$$

The only rule that matches the topsequent to be derived and does not produce duplications is *Unipar*, and one last step of *Sub* finishes the proof search:

$$\frac{\dfrac{a\in l, l=par(l,a), l \,\|\, par(l,a), a\in par(l,a), b\in l, b\in par(l,a)\;\to\; a\in l}{l=par(l,a), l \,\|\, par(l,a), a\in par(l,a), b\in l, b\in par(l,a)\;\to\; a\in l}{\scriptstyle Sub}}{l \,\|\, par(l,a), a\in par(l,a), b\in l, b\in par(l,a)\;\to\; a\in l}{\scriptstyle Unipar}$$

The independence of Euclid's parallel postulate from the other affine axioms is reduced to a proof-theoretical triviality:

Theorem 5.3. *If rule Unipar is deleted from the system of affine geometry, the sequent*

$$b\in l, b\in par(l,a)\;\to\; a\in l$$

is not derivable.

Proof: As above, the last two rules are *Incpar* and *Par*. With rule *Unipar* deleted, no rule matches the sequent to be derived without producing duplications. QED.

6. Related work

In Skolem (1920), a proof-theoretical method is developed for plane projective geometry. Skolem shows that the set of atoms that can be derived from given atoms is finite in projective geometry. To this purpose, he develops a calculus of geometrical rules acting on atomic formulas. Skolem shows as an example the underivability of Desargues' theorem. Ketonen (1944) reformulated Skolem's approach in terms of sequent calculus. In his method, the geometrical axioms appeared as assumptions in the antecedent parts of sequents, making proof analysis very hard. Ketonen extended the method to affine geometry and showed the independence of Euclid's fifth postulate. Except for Ketonen, we have not seen any reference to Skolem's work on the proof theory of geometry before our (1995), even if Skolem's paper was adequately reviewed, the review by Lange-Nielsen appearing between two reviews written by Bernays. The first part of the paper, with the Löwenheim-Skolem theorem, was translated in van Heijenoort's anthology. A part in which Skolem develops a proof theory of lattice theory has been rediscovered recently, as reported in Burris (1995).[1] Ketonen's monograph in turn was extensively presented by Bernays in *JSL* in 1945, but only his discovery of the logical rules of the classical invertible sequent calculus *G3c* has become a permanent part of proof theory so far, through Kleene (1952).

Skolem's work gives a decision method for projective geometry and lattice theory for (in the present terminology) single succedent sequents of the form $P_1, \ldots, P_n \rightarrow Q$, with P_i, Q atomic formulas. An arbitrary quantifier-free formula A is equivalent to the conjunction of a finite number of formulas $P_1 \& \ldots \& P_m \supset Q_1 \vee \ldots \vee Q_n$ with P_i, Q_j atoms. Thus, A is derivable if and only if the sequents $P_1, \ldots, P_m \rightarrow Q_1, \ldots, Q_n$ are derivable. It is shown in Negri, von Plato, and Coquand (2001, theorem 3.2) that if the axioms of a theory are all Harrop formulas and $P_1, \ldots, P_m \rightarrow Q_1, \ldots, Q_n$ is derivable, then $P_1, \ldots, P_m \rightarrow Q_j$ is derivable for some Q_j. The lattice axioms are Harrop formulas, and therefore Skolem's method gives a decision procedure for lattice theory for arbitrary quantifier-free formulas. For projective geometry, it gives a decision method for Harrop formulas. The use of a multisuccedent calculus, as in the present work, is essential for obtaining a full proof-theoretical decision method for non-Harrop theories.

In 1997, Norbert Preining gave a sequent calculus for projective geometry. His starting point was Gentzen's original classical sequent calculus *LK* in which most rules are noninvertible. Moreover, it has explicit rules of weakening and contraction and is therefore not suited to root-first proof analysis. Preining turned the geometrical axioms into rules acting on atoms in the left or right parts of sequents, and established the eliminability of cut by a rather complicated method for which the rule system was tailored.

In our (1998), substantially the same as Negri and von Plato (2001, section 6.6(e)), the constructive axioms of affine geometry of von Plato (1995) were converted into left mathematical rules and the independence of the fifth postulate proved by specific arguments rather than the general result on subterms, theorem 5.2 above. The left and right rule schemes display a duality if relations such as the apartness of two points $a \neq b$ in the antecedent are turned into equalitites in right rules and vice versa. Thus, all results of this paper can be equally read as results on projective and affine geometry formulated as systems of right mathematical rules and intuitionistic basic concepts as in von Plato (1995).

[1] The presentation of Skolem's work in this paper is misleading: Skolem's purely syntactic, proof-theoretical method is described in model-theoretic terms.

As a final remark, we note that it is possible to convert into systems of cut-free rules also axiomatic systems of geometry with existence axioms in place of free parameters and constructions. This possibility follows from the general result of Negri (2001): What are known as "geometric implications" in categorical terminology (no apparent connection with geometry as understood in this paper), convert into systems of mathematical rules with the structural rules admissible.

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