AN APPROACH TO AN ϵ SUBSTITUTION METHOD FOR ID_1

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The draft presented below was given to several colleagues at the ASL meeting in Urbana, June 2000. It summarizes my proposal of a new approach to ID_1 in correspondence with Sergei Tupailo in November 1999. The proposal leaves open an important problem taken up by T. Arai in a manuscript prepared during his stay in Mathematical Institut Mittag-Leffler, Stockholm, 2001. The present proposal outlines a proof separated into several parts as opposed to a global Ackermann-style approach described by T. Arai, where similar parts were mixed. I would like to thank Institute Mittag-Leffler for support of my work and creating very favorable environment during Logic Year 2000/2001.

1 System $ID_1\epsilon$

1.1 Finite ϵ-formulation

There are two sorts of variables: $x, y, z, \ldots$ for natural numbers and $\xi, \eta, \zeta, \ldots$ for ordinals. v will stand for both kinds of variables.

Numerical terms are numerical variables, $0, St$ and $\epsilon x\phi$ for all formulas ϕ. There are many (as much as needed) primitive recursive predicates of numerals and ordinals, including of course $=, <$ and maybe graphs of ordinal functions needed for ordinal notations.

Two binary predicates of ξ, x: I and $I^<$. Abbreviations:

$$I^\xi := \lambda x I(\xi, x); \qquad I^{<\xi} := \lambda x(\exists\eta < \xi)I^\eta(x)$$

$$I^* := I^{<\Omega}$$

No ternary predicate $I(\eta, \xi, x)$ [for $I^{<\xi}$, cf. Arai].

Ordinal terms are ordinal variables, 0 (probably different from numerical 0), Ω, probably results of applying some simple functions and $\epsilon\xi\phi$ for all formulas ϕ. Formulas are "quantifier-free", i.e. are constructed from atomic formulas by Boolean connectives.

Quantifiers are defined from ϵ in a standard way.

Intended interpretation of $\epsilon v \phi$: the least v satisfying ϕ.

1.1.1 Axioms

Tautologies, quantifier free axioms for computable predicates, equality axioms and critical formulas:

$$(\epsilon 0)\ \epsilon v A \neq 0 \to \phi[\epsilon v \phi]$$

$$(\epsilon 1)\ \phi[t] \to \phi[\epsilon v \phi]$$

$$(\epsilon 2)\ \phi[t] \to \epsilon v \phi \leq t$$

$$(\xi)\ t \in I^{\xi} \iff \theta[I^{<\xi}, t]$$

$$(Cl)\ \ t \in I^{\Omega} \to t \in I^{<\Omega}$$

that is,

$$(ID_1)\ \ \phi(I^*, x) \to I^*(x)$$

$$(<\xi)\ t \in I^{<\xi} \iff (\eta_0 < \xi \& t \in I^{\eta_0})$$

where $\eta_0 := \epsilon\eta(\eta < \xi \& t \in I^{\eta})$, that is, ϵ-translation of

$$(<\xi)\ t \in I^{<\xi} \iff (\exists \eta < \xi) t \in I^{\eta})$$

The latter formula is treated as a critical formula for η_0.

We assume that there is exactly one inductive definition in ID_1 defining the predicate I as the least fixed point of the formula θ. $\epsilon 2$ states that $\epsilon v \phi$ is the least number.

1.2 Translation of ID_1 into $ID_1\ \epsilon$

Translate I as I^*.

Lemma 1.1 . *$ID_1\ \epsilon$ derives all axioms of ID_1.*

$TI(<, F)$ is derived from ($\epsilon 1$, $\epsilon 2$) in exactly the same way as ordinary mathematical induction is derived from Hilbert's ϵ-axioms, end section 2 (p.110) of our EA-paper: assume

$$Prog := \forall x((\forall y < x)F[y] \to F[x]).$$

Consider $x_0 := \epsilon x \neg A$ and note $A[x_0] \to \forall x A$ is an $\epsilon 1$-axiom (up to a contraposition). If $x_0 = 0$, then $A[x_0]$, since $A[0]$ follows from $Prog$. If $x_0 \neq 0$ assume $\neg A[x_0]$ for a contradiction. By $Prog$ there is a $y < x_0$ satisfying $\neg A[y]$ which contadicts $\epsilon 2$-axiom for x_0. $\vdash$

Consider

$$ID_2: \quad \forall x(\phi(\lambda x G, x) \to Gx) \to \forall x(Ix \to Gx)$$

that is,

$$Prog(G) \to I \subseteq G$$

$$\dfrac{\dfrac{\xi < \Omega, Prog(G) \Rightarrow I^{<\xi} \subseteq G}{\xi < \Omega, Prog(G), \phi(I^{<\xi}, k) \Rightarrow \phi(\lambda x G, k)}\ mon\ of\ \phi}{\xi < \Omega, Prog(G), I^{\xi}k \Rightarrow \phi(\lambda x G, k)}\ (\xi)$$

$$\dfrac{\dfrac{\dfrac{\dfrac{Gk \Rightarrow Gk \qquad \xi < \Omega, Prog(G), I^{\xi}k \Rightarrow \phi(\lambda x G, k)}{\xi < \Omega, Prog(G), \phi(\lambda x G, k) \to Gk, I^{x}ik \Rightarrow Gk}\ \to\Rightarrow}{\xi < \Omega, Prog(G), I^{\xi}k \Rightarrow Gk}\ Prog(G)}{Prog(G), I^{<\Omega}k \Rightarrow Gk}\ \exists\Rightarrow,\ def\ I^*}{Prog(G) \Rightarrow I^{<\Omega} \subseteq G}\ \Rightarrow\forall$$

1.2.1 Monotonicity

Lemma 1.2 *If $A[X]$ is positive in X then*

$$(\forall x(F[x] - > G[x]) \to \forall x(A[\lambda y F[y]] - > A[\lambda y G[y]])$$

The standard proof goes through, but only for translations of the formulas of ϵ-free quantifier language: all occurrences of X inside ϵ-terms should come from this translation. Now the quantifier case

$$A[X] = \forall z B[z, X]$$

is treated in a standard way. Denoting

$$\epsilon F := \epsilon z B[z, \lambda y F[y]], \qquad \epsilon G := \epsilon z B[z, \lambda y G[y]]$$

we have by IH

$$B[z, \lambda y F[y]] -> B[z, \lambda y G[y]]$$

hence the same with $z/\epsilon G$. (**)

Now assume $\neg B[\epsilon G, \lambda y G[y]]$, then by (**) $\neg B[\epsilon G, \lambda y F[y]]$, and by ϵ-axiom for ϵF, $\neg A[\epsilon F, \lambda y F[y]]$ as required.

1.3 System $ID\infty$.

Derivable objects. *ϵ-substitutitons* or *sequents* consist of components of the form

$$(e, ?), (e, v)$$

where e is an expression and v a value of a suitable type.

If e is a canonical numerical ϵ-term, v is a numeral.

If e is a canonical ordinal ϵ-term, v is an ordinal (notation of our system)

If e is a formula $(n \in I^{\xi})$ or $(n \in I^{<\xi})$ then $v = \top$ (here ξ is an ordinal).

Rules: see below.

We assume as always a fixed system E of critical formulas of the types $\epsilon 1, \epsilon 2, \xi, < \xi$ to be fixed. Formulas $\epsilon 0, Cl$ are taken into consideration in correctness conditions below: $\epsilon 0$ in a standard way, Cl with much more sing and dance.

Correctness conditions: In addition to standard conditions for ϵ-terms $\epsilon x \phi$, one requires for $(\epsilon \xi \phi, \alpha)$ that all axioms $\epsilon 2$ for this $\epsilon \xi \phi$ and α be satisfied. For components $(e \in I', \top)$ the corresponding axiom (ξ) or $(< \xi)$ should be satisfied.

Tentative proposal. $I^{<\xi}$ is always interpreted as

$$\cup_{\eta < \xi} I^{\eta}$$

The last correctness condition to account for (Cl):

$$\text{If } (n \in I^{\Omega}, \top) \text{ is present, then } (n \in I^{\xi}, \top) \text{ is present for some } \xi < \Omega \quad (1)$$

If one of these conditions is violated, the sequent is AxF (after it is computed).

The rules for constructing the initial tree.

Standard computation steps for numerical and ordinal ϵ-terms, and hence Cut for these terms and Axioms AxF,AxH,AxS. Cuts for the formulas $(e \in I')$ with ? playing the role of FALSE in computations. These formulas generate new axioms AxH for critical formulas $\xi, < \xi$. A new infinitary rule (Cl) for components $(n \in I^{\Omega}, \top)$:

$$\frac{...\Gamma, (n \in I^{\Omega}, \top), (n \in I^{\xi}, T) ... \text{ all } \xi < \Omega}{\Gamma, (n \in I^{\Omega}, \top)} \, Cl$$

In fact this is a sound rule, to be replaced by an unsound rule like "locally predicative" I^{ξ}-introduction respecting $<<$ relation, but I'll ignore this now.

These rules are sufficient to construct an initial tree.

1.4 Cut Elimination

As in Pohlers' book [1]. $(n \in I^{<\Omega}, \top)$ plays here the same role as $n \notin I^{<\Omega}$ in Pohlers, and $(n \in I^{<\Omega}, ?)$ plays here the same role as $n \in I^{<\Omega}$ in Pohlers. Ω-cuts are eliminated beginning from the top, so the proof of Pohlers, 27.5 (Impredicative Elimination Lemma) works.

1.5 H-step.

Consider an arbitrary system of closed critical formulas of the form

$$(\epsilon 1), (\epsilon 2), (\xi), (< \xi)$$

The latter is treated as a critical formula for η_0.

H-step for ranks $< \Omega$ is defined in a natural way (cf. Arai).

Consider an H-step for $(n \in I^{\Omega})$. We have for a given ϵ substitution S:

$$|n \in I^{\Omega}|_S = \bot, \quad \text{but} \quad |\theta[I^{<\Omega}, n]|_S = \top \qquad (2)$$

In that case we include not only $(n \in I^\Omega, \top)$, but also $(n \in I^\xi, \top)$ for some $\xi < \Omega$. A computation of ξ includes "a future ordinal of a cut-free derivation of $\theta[I^{<\Omega}, n]$", bounding and collapsing. This ξ serves as a kind of value for Ω in *this* occurrence of $n \in I^\Omega$.

In more detail, (2) implies there is a rather simple derivation

$$S \vdash^\alpha_\rho n \in I^\Omega. \tag{3}$$

Using the correctness condition (1), we reduce (3) to the case when S does not contain components $(m \in I^\Omega, \top)$. Then cut elimination (if $\rho > \Omega$), bounding and collapsing imply

$$S \vdash^\xi_\Omega n \in I^\xi, \qquad \text{for some } \xi < \Omega.$$

This ξ roughly suits the correctness condition (1).

The computation of ξ in the previous paragraph should be refined taking into account the growth of ordinals during elimination of cuts with the rank $>$ Ω. More precisely, for every component $(\epsilon v \phi, ?)$ of the current ϵ substitution S with rank $> \Omega$, we estimate the rank of the trees where $(\epsilon v \phi, ?)$ is replaced by $(\epsilon v \phi, u)$ with $u \neq ?$, and use these estimates to compute ξ. This may require knowledge not only of the current substitution S, but of the whole finite H-process that generated S from the empty substitution $\emptyset$ by H-steps. In other words, ξ is similar to the ordinal Ackermann assigned to an H-process in his termination proof for first order arithmetic.

2 System with TI

A definition of an epsilon-substitution.

Consider a system E of critical formulas $\epsilon 1, \epsilon 2$.

H-step. As always take some false critical formula of the lowest rank.

If it is $\epsilon 1$, do exactly as before: add new value and delete all values of higher rank, retaining values of all remaining ranks. For a critical formula $\epsilon 2$ false in a given ϵ-substitution S choose new value $v :=$

$$\min\{u : \text{ there is a critical formula with a true premise } S\text{-equal to } A[u]\}$$

In other words, do not take the absolute minimum, but only minimum with respect to formulas in given system E of critical formulas. Then do as before, i.e. delete all values of higher rank, retaining values of all remaining ranks.

The reasons should be clear. In the case of arbitrary (natural) well-ordering the bounded minimization operator is not effective anymore, since given element has infinitely many predecessors (=smaller elements). Hence we have to treat $\epsilon 2$ in the same way as we do with ($\epsilon 1$).

Main difference is that now H-step can change a nonzero value, and this was prohibited in all previous formulations beginning with Ackermann. Hilbert-Bernays (in fact of course simply Bernays) specially stress this point.

Nevertheless, the proof for the rank 1 seems to work as before, and the same is true for the non-effective proof. I am not sure which formulation via cut elimination will work: maybe something like B-rule is needed.

References

[1] Pohlers, Wolfram, Proof theory, An introduction, Springer-Verlag, Berlin, 1989, vi+213