

**MORE ON THE EHRENFEUCHT-FRAISSE**

**GAME OF LENGTH  $\omega_1$**

S. SHELAH, T. HYTTINEN

and J. VÄÄNÄNEN

REPORT No. 46, 2000/2001

ISSN 1103-467X

ISRN IML-R- -46-00/01- -SE



**INSTITUT MITTAG-LEFFLER**  
THE ROYAL SWEDISH ACADEMY OF SCIENCES

# More on the Ehrenfeucht-Fraïssé game of length $\omega_1$ .

Tapani Hyttinen *	Saharon Shelah †
Department of Mathematics	Institute of Mathematics
University of Helsinki	Hebrew University
Helsinki, Finland	Jerusalem, Israel

Jouko Väänänen ‡  
Department of Mathematics  
University of Helsinki  
Helsinki, Finland

September 2, 2001

This paper is a continuation of [8]. Let  $\mathfrak{A}$  and  $\mathfrak{B}$  be two first order structures of the same vocabulary  $L$ . We denote the domains of  $\mathfrak{A}$  and  $\mathfrak{B}$  by  $A$  and  $B$  respectively. All vocabularies are assumed to be relational. The *Ehrenfeucht-Fraïssé-game of length  $\gamma$  of  $\mathfrak{A}$  and  $\mathfrak{B}$*  denoted by  $\text{EFG}_\gamma(\mathfrak{A}, \mathfrak{B})$  is defined as follows: There are two players called  $\forall$  and  $\exists$ . First  $\forall$  plays  $x_0$  and then  $\exists$  plays  $y_0$ . After this  $\forall$  plays  $x_1$ , and  $\exists$  plays  $y_1$ , and so on. If  $\langle (x_\beta, y_\beta) : \beta < \alpha \rangle$  has been played and  $\alpha < \gamma$ , then  $\forall$  plays  $x_\alpha$  after which  $\exists$  plays  $y_\alpha$ . Eventually a sequence  $\langle (x_\beta, y_\beta) : \beta < \gamma \rangle$  has been played. The rules of the game say that both players have to play elements of  $A \cup B$ . Moreover, if  $\forall$  plays his  $x_\beta$  in  $A$  ( $B$ ), then  $\exists$  has to play his  $y_\beta$  in  $B$  ( $A$ ). Thus the sequence  $\langle (x_\beta, y_\beta) : \beta < \gamma \rangle$  determines a relation  $\pi \subseteq A \times B$ . Player  $\exists$  wins this round of the game if  $\pi$  is a partial isomorphism. Otherwise  $\forall$

---

\*Partially supported by the Academy of Finland grant #40734.

†Research partially supported by the United States-Israel Binational Science Foundation. Publication number [776]

‡Partially supported by the Academy of Finland grant #40734.

wins. The notion of winning strategy is defined in the usual manner. The game  $\text{EFG}_\gamma^\delta(\mathfrak{A}, \mathfrak{B})$  is defined like  $\text{EFG}_\gamma(\mathfrak{A}, \mathfrak{B})$  except that the players play sequences of length  $< \delta$  at a time. Thus  $\text{EFG}_\gamma(\mathfrak{A}, \mathfrak{B})$  is the same game as  $\text{EFG}_\gamma^2(\mathfrak{A}, \mathfrak{B})$ .

It was proved in [8] that, assuming  $\square_{\omega_1}$ , there are models  $\mathfrak{A}$  and  $\mathfrak{B}$  of cardinality  $\aleph_2$  such that the game  $\mathcal{G}_{\omega_1}(\mathfrak{A}, \mathfrak{B})$  is non-determined. In this paper we weaken the assumption  $\square_{\omega_1}$ , to “ $\omega_2$  is not weakly compact in  $L$ ” (Corollary 8), but we can do this only if we assume CH. We do not know if this is possible without CH. In the other direction, it was proved in [8] that if the  $\omega_1$ -nonstationary ideal on  $\omega_2$  has a  $\sigma$ -closed dense subset, then the game  $\text{EFG}_{\omega_1}(\mathfrak{A}, \mathfrak{B})$  is determined for all  $\mathfrak{A}$  and  $\mathfrak{B}$  of cardinality  $\leq \aleph_2$ . The assumption is equivconsistent with the existence of a measurable cardinal. In this paper we weaken the assumption to a condition which is consistent relative to the existence of a weakly compact cardinal (Corollary 13). Thus we establish:

**Theorem 1** *The following statements are equivconsistent relative to ZFC:*

1. *There is a weakly compact cardinal.*
2. *CH and  $\text{EF}_{\omega_1}(\mathfrak{A}, \mathfrak{B})$  is determined for all models  $\mathfrak{A}$  and  $\mathfrak{B}$  of cardinality  $\aleph_2$ .*

In [8] we proved in ZFC that there are structures  $\mathfrak{A}$  and  $\mathfrak{B}$  of cardinality  $\aleph_3$  with one binary predicate such that the game  $\text{EFG}_{\omega_1}(\mathfrak{A}, \mathfrak{B})$  is non-determined. We now improve this result under some cardinal arithmetic assumptions. We prove:

**Theorem 2** *Assume that  $2^\omega < 2^{\omega_3}$  and  $T$  is a countable complete first order theory. Suppose that one of (i)-(iii) below holds. Then there are  $\mathcal{A}, \mathcal{B} \models T$  of power  $\omega_3$  such that for all cardinals  $1 < \theta \leq \omega_3$ ,  $\text{EF}_{\omega_1}^\theta(\mathcal{A}, \mathcal{B})$  is non-determined.*

- (i)  *$T$  is unstable.*
- (ii)  *$T$  is superstable with DOP or OTOP.*
- (iii)  *$T$  is stable and unsuperstable and  $2^\omega \leq \omega_3$ .*

This result complements the result in [8] that if  $T$  is an  $\omega$ -stable first order theory with NDOP, then  $\text{EFG}_{\omega_1}(\mathfrak{A}, \mathfrak{B})$  is determined for all models  $\mathfrak{A}$  of  $T$  and all models  $\mathfrak{B}$ . This is actually true under the weaker assumption that  $T$  is superstable with NDOP and NOTOP.

**Notation:** We follow Jech [5] in set theoretic notation. We use  $S_n^m$  to denote the set  $\{\alpha < \omega_m : \text{cof}(\alpha) = \omega_n\}$ . Closed and unbounded sets are called cub sets. A set of ordinals is  $\lambda$ -closed if it is closed under supremums of ascending  $\lambda$ -sequences  $\langle \alpha_i : i < \lambda \rangle$  of its elements. A subset of a cardinal is  $\lambda$ -stationary if it meets every  $\lambda$ -closed unbounded subset of the cardinal.

## 1 Getting a weakly compact cardinal

In this section we show that if CH holds and  $\text{EFG}_{\omega_1}(\mathfrak{A}, \mathfrak{B})$  is determined for all models  $\mathfrak{A}$  and  $\mathfrak{B}$  of cardinality  $\aleph_2$ , then  $\omega_2$  is weakly compact in  $L$  (Corollary 8). We use the results from [7] that if  $\omega_2$  is not weakly compact in  $L$ , then there is a bstationary  $S \subseteq S_0^2$  such that for all  $\alpha < \omega_2$  either  $\alpha \cap S$  or  $\alpha \setminus S$  is non-stationary.

If  $I$  is a linear order, we use  $(I)^*$  to denote the reverse order of  $I$ . We call a sequence  $s = (s_\xi)_{\xi < \zeta}$  *cointial sequence of length  $\zeta$*  in  $I$ , if it is decreasing in  $I$  and has no lower bound in  $I$ . The *cointiality*  $\text{coint}(I)$  of a linear order  $I$  is the smallest length of a cointial sequence in  $I$ .

Let  $\theta = \omega + ((\omega_1)^* + \omega) \cdot \omega_1$ .

**Lemma 3** *There is a dense linear order  $I$  such that*

- (i)  $|I| = \aleph_1$ .
- (ii)  $\text{coint}(I) = \aleph_0$
- (iii)  $I \cdot (\alpha + 1) \cong I$  for all  $\alpha \leq \omega_1$ .
- (iv)  $I \cong I \cdot \omega + I \cdot (\omega_1)^*$ .
- (v)  $I \cdot \theta + I \cong I$ .

**Proof.** This is like Lemma 4.7.16 in [9]. If  $J_1$  and  $J_2$  are linear orders, let  $H(J_1, J_2)$  be the set of  $f : n_f \rightarrow J_1 \cup J_2$ , where  $n_f < \omega$  is even,  $f(2i) \in J_1$

and  $f(2i+1) \in J_2$  for all  $i < n_f$ . We can make  $H(J_1, J_2)$  a linear order by ordering the functions lexicographically, i.e.

$$f \leq g \iff \exists m \leq n_f (\forall i < m (f(i) = g(i)) \& (m < n_f \rightarrow f(m) < g(m))).$$

Let  $I_0 = H(\mathbb{Q}, \omega + (\omega_1)^*)$  and  $I_1 = H(I_0, \omega_1)$ . Thus  $I_0 \cong (1 + I_0) \cdot (\omega + (\omega_1)^*) \cdot \mathbb{Q}$  and  $I_1 \cong (1 + I_1) \cdot \omega_1 \cdot I_0$ . By using  $\mathbb{Q} \cong \mathbb{Q} + 1 + \mathbb{Q}$ ,  $\omega = 1 + \omega$  and  $\omega_1 = 1 + \omega_1$ , one gets easily the following, first for  $I_0$ , and then for  $I_1$ :

$$I_0 \cong I_0 + 1 + I_0, \quad I_1 \cong I_1 + 1 + I_1. \quad (1)$$

Let  $I$  be the set of  $f : \omega \rightarrow I_1 \cup \theta$ , where  $f(2i) \in I_1$  and  $f(2i+1) \in \theta$  for all  $i < \omega$  ordered lexicographically. Thus  $I \cong I \cdot \theta \cdot I_1$ . In fact,  $I$  is of the form  $J \cdot \mathbb{Q}$ , so (ii) is true. By (1) and  $\theta \cong 1 + \theta$  one gets immediately (v). As  $I \cong I \cdot \theta \cdot (1 + I_1) \cdot \omega_1 \cdot I_0$ , we get from (v) easily (iii) for  $\alpha = \omega_1$ . From this and  $\alpha + \omega_1 = \omega_1$  we get immediately (iii) for  $\alpha < \omega_1$ . Note that  $\theta \cong \omega + (\omega_1)^* + \theta$ . If we combine this with  $I \cong I \cdot \theta \cdot I_1$  and  $(\omega_1)^* \cong (\omega_1)^* + 1$ , we get (iv).

As to (i), we only have  $|I| = 2^\omega$ . We use this lemma in a context where CH is assumed, so we could simply assume it here. But actually the lemma is true without CH, as we can construct  $I$  in  $L$ . Then  $|I| = \aleph_1$ . Note that our  $I_0$  and  $I_1$  are in  $L$ , and the only property of  $\omega_1$  that we used was that it is a limit ordinal.  $\square$

**Definition 4** Suppose  $S \subseteq S_0^2$ . We define

$$\Phi(S) = \sum_{i < \omega_2} \eta_i,$$

where

$$\eta_i = \begin{cases} I \cdot (\omega_1)^*, & \text{if } i \in S \\ I, & \text{if } i \notin S. \end{cases}$$

Let  $\Phi_{\alpha, \beta}(S)$  be the suborder  $\sum_{\alpha \leq i < \beta} \eta_i$  of  $\Phi(S)$ . The rank of  $x \in \Phi(S)$  is the least  $\alpha$  such that  $x \in \Phi_{\alpha, \alpha+1}(S)$ . We denote this  $\alpha$  by  $\text{rk}(\Phi(S), x)$ .

**Lemma 5** Assume  $S \subseteq S_0^2$  is such that there is no  $\alpha \in S_1^2$  with both  $S \cap \alpha$  and  $(S \cap S_0^2) \setminus S$  stationary. Then

$$\Phi_{\alpha, \beta+1}(S) \cong I$$

whenever  $\alpha < \beta < \omega_2$  and  $\alpha \notin S$ .

**Proof.** This is like Lemma 4.7.19 in [9]. We use Lemma 3 and induction on  $\beta$ .

Let us first assume  $\beta \notin S$ . If  $\beta$  is a successor ordinal, then  $\Phi_{\alpha, \beta+1}(S) \cong I + I = I$  by (iii). If  $\beta$  has cofinality  $\omega$ , then  $\Phi_{\alpha, \beta+1}(S) \cong I \cdot \omega + I \cong I$ . If  $\beta$  has cofinality  $\omega_1$  and  $\beta \cap S$  is non-stationary, then  $I \cong I \cdot \omega_1 + I \cong I$ . Finally, if  $\beta$  has cofinality  $\omega_1$  and  $\beta \setminus S$  is non-stationary, then  $I \cong I \cdot \theta + I \cong I$ , by (v).

Let us then assume  $\beta \in S$ . Thus  $\beta$  has cofinality  $\omega$ . Therefore  $\Phi_{\alpha, \beta+1}(S) \cong I \cdot \omega + I \cdot (\omega_1)^* \cong I$ , by (iv).  $\square$

**Lemma 6** *Assume  $S \subseteq S_0^2$  is such that there is no  $\alpha \in S_1^2$  with both  $S \cap \alpha$  and  $(S \cap S_0^2) \setminus S$  stationary. Then  $\Phi_{0, \alpha}(S) \cong \Phi_{0, \alpha}(\emptyset)$  whenever  $\alpha \in S_1^2$  and  $S \cap \alpha$  is not stationary.*

**Proof.** Let  $(\alpha_\xi)_{\xi < \omega_1}$  be a continuously increasing cofinal sequence in  $\alpha$  such that  $\alpha_\xi \notin S$  for all  $\xi < \omega_1$ . By Lemma 5 there is an isomorphism

$$f_\xi : \Phi_{\alpha_\xi, \alpha_{\xi+1}+1}(S) \rightarrow \Phi_{\alpha_\xi, \alpha_{\xi+1}+1}(\emptyset).$$

Let  $f = \cup_{\xi < \omega_1} f_\xi$ . This is the required isomorphism.  $\square$

**Proposition 7** *Assume CH and that there is  $S \subseteq S_0^2$  such that both  $S$  and  $S_0^2 \setminus S$  are stationary but there is no  $\alpha \in S_1^2$  with both  $S \cap \alpha$  and  $(S \cap S_0^2) \setminus S$  stationary. Then there are models  $\mathfrak{A}$  and  $\mathfrak{B}$  of cardinality  $\aleph_2$  such that  $EF_{\omega_1}(\mathfrak{A}, \mathfrak{B})$  is non-determined.*

**Proof.** We may assume, that  $\{\alpha \in S_1^2 : \alpha \cap S \text{ is non-stationary}\}$  is stationary, for otherwise we work with  $S' = S_0^2 \setminus S$ . Let  $\mathfrak{A} = \Phi(S)$  and  $\mathfrak{B} = \Phi(\emptyset)$ . We first show that  $\exists$  cannot have a winning strategy in  $EF_{\omega+\omega+1}(\mathfrak{A}, \mathfrak{B})$ . Suppose  $\tau$  is a strategy of  $\exists$ . Let  $C$  be the cub of ordinals  $\alpha < \omega_2$  such that if during the first  $\omega$  rounds of the game,  $\forall$  plays elements of the models of rank  $< \alpha$ , then so does  $\exists$  following  $\tau$ . Let  $\delta \in C \cap S$ . Let  $(\delta_n)_{n < \omega}$  be an increasing cofinal sequence in  $\delta$ . Now we let  $\forall$  play against  $\tau$  as follows: On round number  $n < \omega$  we let  $\forall$  play some element of  $\mathfrak{A}$ , if  $n$  is even, and of  $\mathfrak{B}$ , if  $n$  is odd, of rank  $\delta_n$ . During rounds  $\omega + n$ ,  $n < \omega$ , we let  $\forall$  play a cointial sequence of length  $\omega$  in  $\Phi_{\delta, \delta+1}(\emptyset) \subseteq \mathfrak{A}$ . As  $\text{coinit}(\Phi_{\delta, \delta+1}(S)) = \omega_1$ , the game is lost for  $\exists$ . So  $\tau$  could not be a winning strategy.

Suppose then  $\rho$  is a strategy of  $\forall$ . We show that this cannot be a winning strategy. By CH we have an  $\omega_1$ -cub set  $D$  of ordinals  $\delta < \omega_2$  such that if

$\exists$  plays only elements of rank  $< \delta$ , then  $\rho$  directs  $\forall$  to play also elements of rank  $< \delta$  only. Let  $\delta \in D \cap S_1^2$  such that  $\delta \cap S$  is non-stationary. By Lemma 6 there is an isomorphism  $f : \Phi_{0,\alpha}(S) \rightarrow \Phi_{0,\alpha}(\emptyset)$ . Now  $\exists$  can beat  $\rho$  by using  $f$ .  $\square$

**Corollary 8** *If CH holds and  $\text{EFG}_{\omega_1}(\mathfrak{A}, \mathfrak{B})$  is determined for all models  $\mathfrak{A}$  and  $\mathfrak{B}$  of cardinality  $\aleph_2$ , then  $\omega_2$  is weakly compact in  $L$ .*

## 2 Getting determinacy from a weakly compact cardinal

In this section we show that if  $\kappa$  is weakly compact, then there is a forcing extension in which the game  $\text{EFG}_{\omega_1}(\mathfrak{A}, \mathfrak{B})$  is determined for all  $\mathfrak{A}$  and  $\mathfrak{B}$  of cardinality  $\leq \aleph_2$ .

We shall consider models  $\mathfrak{A}, \mathfrak{B}$  of cardinality  $\aleph_2$ , so we may as well assume they have  $\omega_2$  as universe. For such a model  $\mathfrak{A}$  and any ordinal  $\alpha < \omega_2$  we let  $\mathfrak{A}_\alpha$  denote the structure  $\mathfrak{A} \cap \alpha$ . Similarly  $\mathfrak{B}_\alpha$ . Let us first recall the following basic fact from [8]:

**Lemma 9** [8] *Suppose  $\mathfrak{A}$  and  $\mathfrak{B}$  are structures of cardinality  $\aleph_2$ . If  $\forall$  does not have a winning strategy in  $\text{EFG}_{\omega_1}(\mathfrak{A}, \mathfrak{B})$ , then*

$$S = \{\alpha : \mathfrak{A}_\alpha \cong \mathfrak{B}_\alpha\}$$

*is  $\omega_1$ -stationary.*

This shows that to get determinacy of  $\text{EFG}_{\omega_1}(\mathfrak{A}, \mathfrak{B})$  it suffices to give a winning strategy of  $\exists$  under the assumption that the above set  $S$  is  $\omega_1$ -stationary. In [8] an assumption  $I^*(\omega)$  was used. This assumption says that the non- $\omega_1$ -stationary ideal on  $\omega_2$  has a  $\sigma$ -closed dense set. The rough idea was that  $\exists$  uses the Pressing Down Lemma on  $S$  to "normalize" his moves so that he always has an  $\omega_1$ -stationary sets of possible continuations of the game. We use now the same idea. The hypothesis  $I^*(\omega)$  is equiconsistent with a measurable cardinal. Since we assume only the consistency of a weakly compact cardinal, we have to work more.

Suppose  $\kappa$  is a weakly compact cardinal. Let  $\mathcal{I}$  denote the  $\Pi_1^1$ -ideal on  $\kappa$ , i.e.. the ideal of subsets of  $\kappa$  generated by the sets  $\{\alpha : (H(\alpha), \epsilon, A \cap H(\alpha)) \models$

$\neg\phi\}$ , where  $A \subseteq H(\kappa)$  and  $\phi$  is a  $\Pi_1^1$ -sentence such that  $(H(\kappa), \epsilon, A) \models \phi$ . We collapse  $\kappa$  to  $\omega_2$  and then force a cub to the complement of every set  $S \subseteq S_1^2$  in  $\mathcal{I}$ . In the resulting model the above "normalization" strategy of  $\exists$  works even though the non- $\omega_1$ -stationary ideal on  $\omega_2$  may not have a  $\sigma$ -closed dense set.

**Definition 10** *Let  $\mathcal{F}$  be a set of cardinality  $\kappa$  of regressive functions  $\kappa \rightarrow \kappa$  and  $S \subseteq \kappa$ . The game  $\text{PDG}_{\omega_1}(S, \mathcal{F})$  has two players called  $\forall$  and  $\exists$ . They alternately play  $\omega_1$  rounds. During each round  $\forall$  first chooses  $f_i \in \mathcal{F}$ . Then  $\exists$  chooses a subset  $S_i$  of  $\bigcap_{j < i} S_j$  (of  $S$ , if  $i = 0$ ) such that it is unbounded in  $\kappa$  and  $f_i$  is constant on  $S_i$ . Player  $\exists$  wins if he can play all  $\omega_1$  moves following the rules.*

**Lemma 11** *Suppose  $S = \{\alpha < \omega_2 : \alpha \neq 0, \mathfrak{A}_\alpha \cong \mathfrak{B}_\alpha\}$  and  $h_\alpha : \mathfrak{A}_\alpha \cong \mathfrak{B}_\alpha$  for  $\alpha \in S$ . Let*

$$\mathcal{F} = \{f_\alpha : \alpha \in S\} \cup \{g_\alpha : \alpha \in S\},$$

*where  $f_\alpha : \omega_2 \rightarrow \omega_2$  is the regressive function mapping  $\xi$  ( $\neq 0$ ) to  $h_\xi(\alpha)$  if  $\xi > \alpha$ , and to 0 otherwise, and  $g_\alpha$  is the regressive function mapping  $\xi$  ( $\neq 0$ ) to  $(h_\xi)^{-1}(\alpha)$  if  $\xi > \alpha$ , and to 0 otherwise. Suppose  $\exists$  has a winning strategy in  $\text{PDG}_{\omega_1}(S, \mathcal{F})$ . Then  $\exists$  has a winning strategy in the game  $\text{EFG}_{\omega_1}^{\aleph_2}(\mathfrak{A}, \mathfrak{B})$ .*

**Proof.** We present the proof for  $\text{EFG}_{\omega_1}^2(\mathfrak{A}, \mathfrak{B})$ . The case of  $\text{EFG}_{\omega_1}^{\aleph_2}(\mathfrak{A}, \mathfrak{B})$  is similar.  $\mathcal{H} = \{h_\alpha : \alpha \in S\}$ , where  $h_\alpha : \mathfrak{A}_\alpha \cong \mathfrak{B}_\alpha$  for  $\alpha \in S$ . Let  $\tau$  be a winning strategy of  $\exists$  in the game  $\text{PDG}_{\omega_1}(S, \mathcal{F})$ . Suppose the sequence  $\langle (x_i, y_i) : i < \alpha \rangle$  has been played, where  $\alpha < \omega_1$ ,  $x_i$  denotes a move of  $\forall$  and  $y_i$  a move of  $\exists$ . Suppose  $\forall$  plays next  $x_\alpha$ . During the game  $\exists$  also plays  $\text{PDG}_{\omega_1}(S, \mathcal{F})$ . Let us denote his moves in  $\text{PDG}_{\omega_1}(S, \mathcal{F})$  by  $S_i$ . Thus  $S_j \subseteq S_i$  for  $i < j < \alpha$ . The point of the sets  $S_i$  is that  $\exists$  has taken care that for all  $i < \alpha$  and  $j \in S_i$  we have  $y_i = h_j(x_i)$  or  $x_i = h_j(y_i)$  depending on whether  $x_i \in \mathfrak{A}$  or  $x_i \in \mathfrak{B}$ . Let  $S'_\alpha = \bigcap_{i < \alpha} S_i \setminus \alpha_i$ . The winning strategy  $\tau$  gives an  $S_\alpha \subseteq S'_\alpha$  and a  $y_\alpha$  such that  $f_i(x_\alpha) = y_\alpha$  for all  $i \in S_\alpha$ , if  $x_\alpha \in \mathfrak{A}$ , and  $g_i(x_\alpha) = y_\alpha$  for all  $i \in S_\alpha$ , if  $x_\alpha \in \mathfrak{B}$ . This element  $y_\alpha$  is the next move of  $\exists$ . Using this strategy  $\exists$  cannot lose and hence wins.  $\square$

**Theorem 12** *It is consistent relative to the consistency of a weakly compact cardinal, that for every  $\omega_1$ -stationary  $S \subseteq \omega_2$  and every set  $\mathcal{F}$  of cardinality  $\aleph_2$  of regressive functions  $\omega_2 \rightarrow \omega_2$ ,  $\exists$  has a winning strategy in the game  $\text{PDG}_{\omega_1}(S, \mathcal{F})$ .*

**Proof.** We may assume GCH. Suppose  $\kappa$  is weakly compact. Let  $\mathbb{Q}$  be the Levy-collapse of  $\kappa$  to  $\aleph_2$ . In  $V^{\mathbb{Q}}$  we define by induction a sequence  $\mathbb{P}_\alpha$ ,  $\alpha < \kappa^+$ , of forcing notions. Let  $(A_\alpha)$ ,  $\alpha < \kappa^+$ , be a complete list of all sets in the  $\Pi_1^1$ -ideal  $\mathcal{I}$  on  $\kappa$  such that every element of  $A_\alpha$  has uncountable cofinality. If  $\alpha$  is limit of cofinality  $\leq \omega_1$ , then  $\mathbb{P}_\alpha$  is the inverse limit of all  $\mathbb{P}_\beta$ ,  $\beta < \alpha$ . For other limit  $\alpha$ ,  $\mathbb{P}_\alpha$  is the direct limit of  $\mathbb{P}_\beta$ ,  $\beta < \alpha$ . At successor stages we let  $\mathbb{P}_{\alpha+1} = \mathbb{P}_\alpha \star \mathbb{R}_\alpha$ , where  $\mathbb{R}_\alpha$  is defined as follows:  $q \in \mathbb{R}_\alpha$  iff  $q$  is a bounded closed sequence of elements of  $\kappa$  such that  $q \cap A_\alpha = \emptyset$ .  $\mathbb{R}_\alpha$  is ordered by the end extension relation. Thus each  $\mathbb{P}_\alpha$  is countably closed. Let  $\mathbb{P} = \mathbb{P}_{\kappa^+}$ . Now  $\mathbb{Q} \star \mathbb{P}$  satisfies the  $\kappa^+$ -chain condition. Note also that for all  $\alpha < \kappa^+$ ,  $\mathbb{Q} \star \mathbb{P}_\alpha$  has power  $\kappa$ . We prove that it is true in  $V^{\mathbb{Q}}$  that  $\mathbb{P}_\alpha$  does not add new subsets of  $\kappa$  of cardinality  $\leq \aleph_1$ , hence  $\kappa$  remains  $\aleph_2$  also after forcing with  $\mathbb{P}$ . It follows also that  $\mathbb{Q} \star \mathbb{P}$  and each  $\mathbb{Q} \star \mathbb{P}_\alpha$  are countably closed.

We show now that in  $V^{\mathbb{Q} \star \mathbb{P}}$  the claim is true. Suppose  $S$  and a set  $\mathcal{F} = \{f_\alpha : \alpha < \kappa\}$ , of regressive functions  $\kappa \rightarrow \kappa$  are given in  $V^{\mathbb{Q} \star \mathbb{P}}$  such that (in  $V^{\mathbb{Q} \star \mathbb{P}}$ )  $S \subseteq S_1^2$  is  $\omega_1$ -stationary. Suppose  $\alpha < \kappa^+$  is such that  $\tilde{S}, \tilde{\mathcal{F}}$  and  $\tilde{f}_i$  are  $\mathbb{Q} \star \mathbb{P}_\alpha$ -names for  $S, \mathcal{F}$  and  $f_i$ , correspondingly. Since  $S$  is  $\omega_1$ -stationary in  $V^{\mathbb{Q} \star \mathbb{P}}$ ,  $S$  is not in the ideal generated by  $\mathcal{I}$  in  $V^{\mathbb{Q} \star \mathbb{P}_\alpha}$ . Suppose  $(p, q) \Vdash \tilde{S} \notin \mathcal{I}$ . For a contradiction, suppose also that  $(p, q)$  forces that  $\exists$  does not have a winning strategy in the game  $BM_{\omega_1}(S, \mathcal{F})$ .

Let  $(\mathcal{B}'_0, \in)$  be a sufficiently elementary substructure of  $(V, \in)$  such that  $|\mathcal{B}'_0| = \kappa$ ,  $\mathcal{B}'_0^{<\kappa} \subseteq \mathcal{B}'_0$ ,  $\mathbb{Q}, \mathbb{P}_\alpha, \alpha, \kappa, \tilde{\mathcal{F}}, \tilde{f}_i, \tilde{S}$ , are in  $\mathcal{B}'_0$ , and  $\alpha \cup \kappa \subseteq \mathcal{B}'_0$ . Let  $\mathcal{B}_0$  be the transitive collapse of  $\mathcal{B}'_0$ . Thus  $\mathbb{Q}, \mathbb{P}_\alpha, \alpha, \kappa \in \mathcal{B}_0$ ,  $A_j \in \mathcal{B}_0$  for  $i \leq \alpha$  and  $\tilde{f}_i \in \mathcal{B}_0$  for  $i < \kappa$ . Let

$$T = \{\alpha < \kappa \mid \exists (p', q') \leq (p, q) ((p', q') \Vdash_{\mathbb{Q} \star \mathbb{P}_\alpha} \alpha \in \tilde{S})\}.$$

Clearly  $T \in \mathcal{B}_0$  and  $T \notin \mathcal{I}$ . By weak compactness, there are a transitive  $\mathcal{B}_1$  and an elementary embedding  $j : \mathcal{B}_0 \rightarrow \mathcal{B}_1$  such that  $\kappa$  is the critical point of  $j$ ,  $\kappa \in j(T)$  and  $\kappa \notin j(A_i)$  for  $i \leq \alpha$ . So there is some  $(p', q') \in j(\mathbb{Q} \star \mathbb{P}_\alpha)$  such that  $(p', q') \leq j((p, q)) = (p, q)$  and  $(p', q') \Vdash_{j(\mathbb{Q} \star \mathbb{P}_\alpha)} \kappa \in j(\tilde{S})$ . Note that  $\mathbb{Q}, \mathbb{P}_\alpha \in \mathcal{B}_1$  and  $\tilde{f}_i \in \mathcal{B}_1$  for  $i < \kappa$ .

By (the proof of) Lemma 3 in [7], there are a  $\mathbb{Q} \star \mathbb{P}_\alpha$ -generic  $G$  over  $\mathcal{B}_1$  and a forcing notion  $\mathbb{R} \in \mathcal{B}_1[G]$  such that  $(p, q) \in G$ , in  $\mathcal{B}_1[G]$ ,  $\mathbb{R}$  is countably closed, for all  $\mathbb{R}$ -generic  $K$  over  $\mathcal{B}_1[G]$ , there is a canonical  $j(\mathbb{Q} \star \mathbb{P}_\alpha)$ -generic  $G_K$  over  $\mathcal{B}_1$  such that  $\mathcal{B}_1[G_K] = \mathcal{B}_1[G][K]$  and for some  $K$ ,  $G_K$  is such that  $(p', q') \in G'$ . Then for every  $\mathbb{Q} \star \mathbb{P}_\alpha$ -name  $\tilde{X} \in \mathcal{B}_0$ , there is a canonical  $\mathbb{R}$ -name  $\tilde{Y} \in \mathcal{B}_1[G]$  such that for all  $\mathbb{R}$ -generic  $K$  over  $\mathcal{B}_1[G]$ ,  $j(\tilde{X})$  and  $\tilde{Y}$

have the same interpretation in  $\mathcal{B}_1[G][K]$ . We do not distinguish  $j(\tilde{X})$  and  $\tilde{Y}$ . With this notation, there is  $r \in \mathbb{R}$  which forces in  $\mathcal{B}_1[G]$ , that  $\kappa \in j(\tilde{S})$ . Then there is some  $(p^*, q^*) \leq (p, q)$  in  $G$  that in  $\mathcal{B}_1$  forces the existence of such  $\mathbb{R}$  and  $r$ . So we may assume that  $G$  is generic over  $V$  and our  $V^{\mathbb{Q} \star \mathbb{P}_\alpha}$  is the same as  $V[G]$ .

We describe in  $\mathcal{B}_1[G]$  a winning strategy of  $\exists$  in the game  $BM_{\omega_1}(S, \mathcal{F})$ . This is a contradiction since all possible winning plays of  $\forall$  are in  $\mathcal{B}_1[G]$  and being unbounded is absolute in transitive models. The strategy of  $\exists$  is to play on the side conditions  $q^i$  in  $\mathcal{B}_1[G]$  and sets  $S_i \in \mathcal{B}_0[G]$  with  $\mathbb{Q} \star \mathbb{P}_\alpha$ -names  $\tilde{S}_i$  in  $\mathcal{B}_0$  such that

1.  $q^i \in \mathbb{R}$ .
2.  $q^0 \leq r$ .
3.  $i < k < \omega_1$  implies  $q^k \leq q^i$ .
4.  $i < k < \omega_1$  implies  $S_k \subseteq S_i \subseteq S$ .
5.  $q^i \Vdash_{\mathbb{R}} \kappa \in j(\tilde{S}_i)$  in  $\mathcal{B}_1[G]$ .

Suppose  $\exists$  has followed this strategy, forming conditions  $q^i$  and sets  $S_i$  for  $i < k$ . Let  $p = \inf(\{q^i : i < k\})$ . If we let  $S$  to be  $\bigcap_{i < k} S_i$  and  $\tilde{S}$  a name for this, then in  $\mathcal{B}_1[G]$ ,

$$p \Vdash_{\mathbb{R}} \kappa \in j(\tilde{S}).$$

Suppose then  $\forall$  moves  $f_k \in \mathcal{F}$ . Let  $q^k \leq p$  such that for some  $\delta < \kappa$  we have  $q^k \Vdash_{\mathbb{R}} j(\tilde{f}_k)(\kappa) = \delta$  in  $\mathcal{B}_1[G]$  and let  $S_k$  be  $\{\beta \in S : f_k(\beta) = \delta\}$  and  $\tilde{S}_k$  a name for this. Then  $q^k \Vdash_{\mathbb{R}} \kappa \in j(\tilde{S}_k)$  in  $\mathcal{B}_1[G]$ .

Finally we have to prove that  $\mathbb{Q} \star \mathbb{P}_\alpha$  does not add new subsets of  $\kappa$  of cardinality  $\leq \aleph_1$  over and above those added by  $\mathbb{Q}$ . The proof of this is, mutatis mutandis, like the proof of the Main fact (page 761) in [7]. Here we use the assumption  $\kappa \notin j(A_i)$  for  $i \leq \alpha$ . Thus, if  $C$  is a generic sequence in the complement of  $j(A_\beta)$  in  $V^{j(\mathbb{Q} \star \mathbb{P}_\alpha)}$ , then we can continue it to a closed condition  $C \cup \{\kappa\} \in \mathbb{R}_{j(\beta)}$ .  $\square$

Results similar to Theorem 12 have been treated also in [13] and [14].

**Corollary 13** *It is consistent relative to the consistency of a weakly compact cardinal, that the game  $\text{EFG}_{\omega_1}(\mathfrak{A}, \mathfrak{B})$  is determined for all  $\mathfrak{A}$  and  $\mathfrak{B}$  of cardinality  $\leq \aleph_2$ .*

### 3 Non-determinacy and structure theory

In this section we prove Theorem 2, which essentially establishes, under cardinality assumptions concerning the continuum, the existence of non-determined Ehrenfeucht-Fraïssé games of length  $\omega_1$  for models of *non-classifiable* theories. This complements the observation, made in [8], that the Ehrenfeucht-Fraïssé game of length  $\omega_1$  is determined for models of *classifiable* theories.

We start by proving Theorem 2 under assumption (iii), which we consider the most interesting case. That is, we start with a countable complete stable and unsuperstable first order theory and show that, assuming  $2^\omega \leq \omega_3$ , it has two models  $\mathfrak{A}$  and  $\mathfrak{B}$  of cardinality  $\aleph_3$  for which  $\text{EFG}_{\omega_1}(\mathfrak{A}, \mathfrak{B})$  is non-determined. Actually, we construct  $\mathfrak{A}$  and  $\mathfrak{B}$  so that  $\exists$  does not have a winning strategy even in  $\text{EFG}_{\omega+\omega}^2(\mathfrak{A}, \mathfrak{B})$  and  $\forall$  does not have a winning strategy even in  $\text{EFG}_{\omega_1}^{\omega_3}(\mathfrak{A}, \mathfrak{B})$ .

We then prove Theorem 2 under assumption (i), that is, we now start with a countable complete unstable first order theory and show that, assuming  $2^\omega < 2^{\omega_3}$ , it has two models  $\mathfrak{A}$  and  $\mathfrak{B}$  of cardinality  $\aleph_3$  for which  $\text{EFG}_{\omega_1}(\mathfrak{A}, \mathfrak{B})$  is non-determined.

Theorem 2 under assumption (ii) can be dealt with in the same way as under assumption (i). The section ends with some remarks on possible improvements.

#### 3.1 The stable unsuperstable case

We will prove Theorem 2, case (iii), in a series of lemmas. We assume  $\omega_3^\omega = \omega_3$  all the time. Let  $T$  be a countable complete stable and unsuperstable first order theory. As usual, we work inside a large saturated model  $\mathbf{M}$  of  $T$ . We start by fixing some notation. By a tree  $I$  we mean a lexicographically ordered downwards closed subtree of  $\theta^{<(\omega+1)}$  for some linear order  $\theta$ , that is,  $I = (I, \ll, P_\alpha, <, H)_{\alpha \leq \omega} \in K_{tr}^\omega(\theta)$ , see [4] Definition 8.2 or [11]. For a while, we fix a tree  $I \in K_{tr}^\omega(\lambda)$ , where  $\lambda$  is some large enough cardinal, so that  $(I, \ll)$  is isomorphic to  $\lambda^{<(\omega+1)}$ . As in [3], for  $u, v \in \mathcal{P}_\omega(I)$  (=finite subsets of  $I$ ), we define  $r(u, v)$  to be the unique set  $R$  which satisfies

- (I)  $R \subseteq X_{u,v} = \{H(\eta, \xi) \mid \eta \in u, \xi \in v\}$ ,
- (II) For all  $\nu \in X_{u,v} - R$ , there is  $\nu' \in R$  such that  $\nu \ll \nu'$ ,
- (III) If  $\eta$  and  $\xi$  are distinct elements of  $R$ , then  $\eta \not\ll \xi$ .

We write  $u \leq v$  if  $r(u, v) = r(u, u)$ . For more on these definitions, see [3]. In [3], it is shown that there are models  $\mathcal{A}$  and  $\mathcal{A}_u$ ,  $u \in \mathcal{P}_\omega(I)$ , and sequences  $a_\eta$  from  $\mathcal{A}_{\{\eta\}}$ ,  $\eta \in I$ , such that

- (i)  $\mathcal{A} = \bigcup_{u \in \mathcal{P}_\omega(I)} \mathcal{A}_u \models T$ ,
- (ii) if  $u \leq v$ , then  $\mathcal{A}_u \subseteq \mathcal{A}_v$ ,
- (iii) for all  $u, v \in \mathcal{P}_\omega(I)$ ,  $\mathcal{A}_u \downarrow_{\mathcal{A}_{r(u,v)}} \mathcal{A}_v$ ,
- (iv) for all  $u \in \mathcal{P}_\omega(I)$ ,  $|\mathcal{A}_u| \leq \omega_3$ ,
- (v) if  $P_\omega(\eta)$  holds and  $\xi \ll \eta$  is an immediate successor of  $\xi'$ , then

$$a_\eta \not\downarrow_{\mathcal{A}_{\{\xi'\}}} a_\xi.$$

These models are exactly what we want except that they are too large, we want the models  $\mathcal{A}_u$ ,  $u \in \mathcal{P}_\omega(I)$ , to be countable. In order to get this, we use the Ehrenfeucht-Mostowski construction.

We extend the signature  $L$  of  $T$  to  $L_*$  by adding  $\omega_3$  new function symbols, some of which will be interpreted in  $\mathbf{M}$  so that they provide Skolem-functions for the  $L$ -formulas. In addition we interpret the functions so that if we write  $SH_*(u)$  for the  $L_*$ -Skolem-hull of  $\{a_\eta \mid \{\eta\} \leq u\}$  then

- (vi) for all  $u \in \mathcal{P}_\omega(I)$ ,  $SH_*(u) = \mathcal{A}_u$ .

By the usual argument (using [11, Appendix Theorem 2.6] and compactness) we can interpret the new function symbols so that  $\mathbf{M}$  remains sufficiently saturated and the following holds

- (vii) if  $U$  is a downwards closed subtree of  $I$  and  $f$  is an automorphism of  $U$ , then there is an  $L_*$ -automorphism  $g$  of  $\bigcup_{u \in \mathcal{P}_\omega(U)} \mathcal{A}_u$  such that for all  $\eta \in U$ ,  $g(a_\eta) = a_{f(\eta)}$ .

Finally, it is easy to see that we can choose countable  $L_1 \subseteq L_*$  so that  $L \subseteq L_1$ ,  $L_1$  contains the Skolem-functions for the  $L$ -formulas and if we write  $SH_1(u)$  for the  $L_1$ -Skolem-hull of  $\{a_\eta \mid \{\eta\} \leq u\}$  then

- (viii) for all  $u, v \in \mathcal{P}_\omega(I)$ ,  $SH_1(u) \downarrow_{SH_1(v)} SH_*(v)$ .

So we have proved the following lemma (for the notion  $\Phi$  proper for  $K_{tr}^\omega$  and the Ehrenfeucht-Mostowski models  $EM^1(J, \Phi)$ , see [4] Definition 8.1 or [11]).

**Lemma 14** *There are countable  $L_1 \supseteq L$  and  $\Phi$  proper for  $K_{tr}^\omega$  such that the following holds:*

- (a) *For all  $J \in K_{tr}^\omega$  there are an  $L_1$ -model  $EM^1(J, \Phi) \models T$  and sequences  $a_\eta \in EM^1(J, \Phi)$ ,  $\eta \in J$ , such that  $EM^1(J, \Phi)$  is the  $L_1$ -Skolem-hull of  $\{a_\eta \mid \eta \in J\}$  (i.e.  $\{a_\eta \mid \eta \in J\}$  is the skeleton of  $EM^1(J, \Phi)$  and as before for  $u \subseteq J$ ,  $SH_1(u)$  denotes the  $L_1$ -Skolem hull of  $\{a_\eta \mid \{\eta\} \leq u\}$ ).*
- (b) *If  $U$  is a downwards closed subtree of  $J$  and  $f$  is an automorphism of  $U$ , then there is an  $L_1$ -automorphism  $g$  of  $SH_1(U)$  such that for all  $\eta \in U$ ,  $g(a_\eta) = a_{f(\eta)}$ .*
- (c) *Assume  $(\eta_i)_{i < \omega}$  is a strictly  $\ll$ -increasing sequence of elements of  $J$ ,  $\eta_{i+1}$  is an immediate successor of  $\eta_i$  and  $\eta_0$  is the root. Then  $(\eta_i)_{i < \omega}$  has an upper bound in  $J$  iff there is a sequence  $a \in EM(J, \Phi)$  such that for all  $i < \omega$ ,  $a \restriction_{SH_1(\{\eta_i\})} a_{\eta_{i+1}}$ .  $\square$*

We will write  $EM(J, \Phi)$  for  $EM^1(J, \Phi) \upharpoonright L$ .

Our next goal is to define the skeletons for the models  $\mathcal{A}$  and  $\mathcal{B}$  in the theorem. For this we use the weak box from [8]. By  $S_m^n$  we denote the set  $\{\alpha < \omega_n \mid cf(\alpha) = \omega_m\}$ .

**Theorem 15** ([8, Lemma 16]) *There are sets  $S$ ,  $U$  and  $C_\alpha$ ,  $\alpha \in S$ , such that the following holds:*

- (a)  $S \subseteq S_0^3 \cup S_1^3$  and  $S \cap S_1^3$  is stationary,
- (b)  $U \subseteq S_0^3$  is stationary and  $S \cap U = \emptyset$ ,
- (c) for all  $\alpha \in S$ ,  $C_\alpha \subseteq \alpha \cap S$  is closed in  $\alpha$  and of order-type  $\leq \omega_1$ ,
- (d) for all  $\alpha \in S$ , if  $\beta \in C_\alpha$ , then  $C_\beta = C_\alpha \cap \beta$ ,
- (e) for all  $\alpha \in S \cap S_1^3$ ,  $C_\alpha$  is unbounded in  $\alpha$ .  $\square$

We will construct trees  $I_\alpha$  and  $J_\alpha$ ,  $\alpha < \omega_3$ , so that the following holds:

- (1) if  $\alpha < \beta$  then  $I_\alpha$  is a submodel of  $I_\beta$  and  $J_\alpha$  is a submodel of  $J_\beta$ ; now for  $\eta \in I_\alpha$ , we will write  $rk(\eta)$  for the least  $\beta$  such that  $\eta \in I_\beta$  and similarly for  $\eta \in J_\alpha$ ,
- (2) for all  $\alpha \in S$ , there is an isomorphism  $G_\alpha : I_\alpha \rightarrow J_\alpha$ ,
- (3) if  $\alpha \in C_\beta$ , then  $G_\alpha \subseteq G_\beta$ ,

- (4) for all  $\alpha \leq \beta$  and  $\eta \in I_\alpha$ , if  $P_\omega(\eta)$  does not hold, then there is an immediate successor  $\xi$  of  $\eta$  such that  $\xi \in I_{\beta+1} - I_\beta$
- (5) if  $(\eta_i)_{i < \omega}$  is an increasing sequence of elements of  $I_\alpha$  (for some  $\alpha$ ) and the sequence has an upper bound  $\xi$  in  $I_\alpha$ , then  $rk(\xi) = \sup_{i < \omega} rk(\eta_i)$  and similarly for sequences from  $J_\alpha$ ,
- (6) if  $(\eta_i)_{i < \omega}$  is an increasing sequence of elements of  $I_\alpha$ ,  $(rk(\eta_i))_{i < \omega}$  is not eventually constant and the sequence has an upper bound  $\xi$  in  $I_\alpha$ , then  $rk(\xi) (= \sup_{i < \omega} rk(\eta_i)) \in U$ ; in  $J_\alpha$  such sequences never have an upper bound,
- (7)  $|I_\alpha| \leq \omega_3$  and  $|J_\alpha| \leq \omega_3$ ,
- (8)  $I_\alpha, J_\alpha \subseteq H_\omega(\omega_3)$ , where  $H_\omega(\omega_3)$  is the least set  $H$  such that  $\omega_3 \subseteq H$  and if  $E \subseteq H$  is of power  $\leq \omega$ , then  $E \in H$ .

It is easy to see that such trees can be constructed by induction on  $\alpha$ . However, in order to get what we want we need to do a bit more work when we define  $I_\alpha$  and  $J_\alpha$  in the case  $\alpha \in U$ . In order to decide, which branches like the one in (6) above, we want to have an upper bound, we use a guessing machine from [12] called black box, which we formulate so that it fits exactly to our purposes.

**Theorem 16** ([12]) ( $\omega_3^\omega = \omega_3$ .) *There are  $(\overline{M}^\alpha, \eta^\alpha)$ ,  $\alpha < \omega_3$ , such that*

- (i)  $\overline{M}^\alpha = (M_i^\alpha)_{i < \omega}$  is an increasing elementary chain of elementary submodels of some  $(H_\omega(\omega_3), A, B, \sigma)$ , such that  $A, B \subseteq H_\omega(\omega_3)$  and  $\sigma$  is a strategy of  $\exists$  in  $EF_\omega^2(A, B)$  ( $A$  and  $B$  can be viewed as models of the empty signature),
- (ii)  $M_i^\alpha = (M_i^\alpha, A_i^\alpha, B_i^\alpha, \sigma_i^\alpha) \in H_\omega(\omega_3)$ ,
- (iii)  $\eta^\alpha$  is an increasing function from  $\omega$  to  $\omega_3$ ,  $M_i^\alpha \in H_\omega(\eta^\alpha(i+1))$  and  $\sup_{i < \omega} \eta^\alpha(i) \in U$ ,
- (iv)  $(\eta^\alpha(j))_{j \leq i}, (M_j^\alpha)_{j \leq i} \in M_{i+1}^\alpha$ ,
- (v) if  $\alpha \neq \beta$ , then  $\eta^\alpha \neq \eta^\beta$ ,

(vi) *player I does not have a winning strategy for the following game: The length of the game is  $\omega$ . At each move  $i < \omega$ , first I chooses  $M_i$  and then II chooses  $\alpha_i < \omega_3$ . I must play so that in the end (i), (ii) and (iv) above are satisfied. I wins if he has played according to the rules and there is no  $\alpha < \omega_3$  such that  $((M_i)_{i < \omega}, (\alpha_i)_{i < \omega}) = (\overline{M}^\alpha, \eta^\alpha)$ .*

First we uniformize (partially) the Ehrenfeucht-Mostowski construction: We assume that for all  $I, I' \in K_{tr}^\omega$ , if  $I$  is a substructure of  $I'$  and  $I' \subseteq H_\omega(\omega_3)$ , then there is a unique model  $EM^1(I, \Phi)$ , it is a substructure of  $EM^1(I', \Phi)$  and  $EM^1(I', \Phi) \subseteq H_\omega(\omega_3)$ .

So let  $\alpha \in U$  and assume that  $I_\beta$  and  $J_\beta$  are defined for all  $\beta < \alpha$ . Write  $I_\alpha^* = \cup_{\beta < \alpha} I_\beta$  and  $J_\alpha^* = \cup_{\beta < \alpha} J_\beta$ . For  $\gamma < \omega_3$ , we write  $M^\gamma$  for  $\cup_{i < \omega} M_i^\gamma$  and  $A^\gamma, \mathcal{B}^\gamma$  and  $\sigma^\gamma$  are defined similarly. Let  $W^\alpha$  be the set of all  $\gamma < \omega_3$  such that

- (a)  $A^\gamma = EM(I_\alpha^* \cap M^\gamma, \Phi)$  and  $B^\gamma = EM(J_\alpha^* \cap M^\gamma, \Phi)$ ,
- (b)  $\sup_{i < \omega} \eta^\gamma(i) = \alpha$ ,
- (c) there are  $\xi_i^\gamma \in I_\alpha^* \cap M^\gamma$ ,  $i < \omega$ , such that  $\xi_0^\gamma$  is the root of  $I_\alpha^*$ ,  $\xi_{i+1}^\gamma$  is an immediate successor of  $\xi_i^\gamma$  and  $\xi_i^\gamma \in I_{\eta^\gamma(i)+1} - I_{\eta^\gamma(i)}$ .

Notice that by Theorem 16 (v), if  $\gamma \neq \delta$ , then  $(\xi_i^\gamma)_{i < \omega} \neq (\xi_i^\delta)_{i < \omega}$ . Let  $C_i^\gamma = SH(\{\xi_i^\gamma\})$ . Then we can find a partial function  $g^\gamma : A^\gamma \rightarrow \mathcal{B}^\gamma$  such that

- (d)  $\text{dom}(g^\gamma) = \cup_{i < \omega} C_i^\gamma$ ,
- (e)  $g^\gamma$  is a result of a play of  $EF_\omega^2(A^\gamma, B^\gamma)$  in which  $\exists$  has used  $\sigma^\gamma$ .

We let  $W_J^\alpha$  be the set of those  $\gamma \in W^\alpha$  such that

- (f)  $g^\gamma$  is a partial isomorphism from  $EM(I_\alpha^* \cap M^\gamma, \Phi)$  to  $EM(J_\alpha^\gamma \cap M^\gamma, \Phi)$ ,
- (g) there is  $J$  such that if we let  $J_\alpha = J$ , then (1),(5)-(8) above are satisfied and there is a sequence  $a \in EM(J, \Phi)$  such that for all  $i < \omega$ ,  $a \restriction_{g^\gamma(C_i^\gamma)} = g^\gamma(a \restriction_{\xi_{i+1}^\gamma})$ .

We let  $W_I^\alpha$  be the set of all  $\gamma \in W^\alpha - W_J^\alpha$  such that  $g^\gamma$  satisfies (f) above.

Now we can define  $I_\alpha$  and  $J_\alpha$ . First we choose  $I_\alpha$  so that it consists of all  $\eta \in I_\alpha^*$  together with the supremums for the branches  $(\xi_i^\gamma)_{i < \omega}$ ,  $\gamma \in W_I^\alpha$ .  $J_\alpha$  is chosen so that it satisfies (g) for all  $\gamma \in W_J^\alpha$  (and so especially (1),(5)-(8)).

Then we let  $I = \cup_{\alpha < \omega_3} I_\alpha$ ,  $J = \cup_{\alpha < \omega_3} J_\alpha$ ,  $\mathcal{A} = EM(I, \Phi)$  and  $\mathcal{B} = EM(J, \Phi)$ . Clearly  $\mathcal{A}$  and  $\mathcal{B}$  can be chosen so that  $\mathcal{A}, \mathcal{B} \subseteq H_\omega(\omega_3)$ .

**Lemma 17**  $\forall$  does not have a winning strategy for  $EF_{\omega_1}^{\omega_3}(\mathcal{A}, \mathcal{B})$ .

**Proof.** For this it is enough to show that  $A$  does not have a winning strategy for  $EF_{\omega_1}^{\omega_3}(I, J)$ , which is clear by (2) and (3) above and Theorem 15.  $\square$

**Lemma 18**  $\exists$  does not have a winning strategy for  $EF_{\omega+\omega}^2(\mathcal{A}, \mathcal{B})$ .

**Proof.** For a contradiction, assume  $\sigma$  is a winning strategy of  $\exists$  for the game  $EF_{\omega+\omega}^2(\mathcal{A}, \mathcal{B})$ . We play a round of the game defined in Theorem 16 (vi). We let player I play so that he follows the rules and

- (i) for all  $i < \omega$ ,  $M_i \prec (H_\omega(\omega_3), \mathcal{A}, \mathcal{B}, \sigma \upharpoonright \omega)$ ,
- (ii) for all  $\delta, \delta' \in M_i$ , if  $\delta \leq \delta'$ ,  $\eta \in I_\delta \cap M_i$  and  $P_\omega(\eta)$  does not hold, then there is  $\xi \in (I_{\delta'+1} - I_\delta) \cap M_{i+1}$  such that  $\xi$  is an immediate successor of  $\eta$ ,
- (iii) the Skolem-hulls of  $\{a_\eta \mid \eta \in I \cap M_i\}$  and  $\{a_\eta \mid \eta \in J \cap M_i\}$  are subsets of  $M_{i+1}$ ,
- (iv)  $\mathcal{A} \cap M_i$  is a subset of the Skolem hull of  $\{a_\eta \mid \eta \in I \cap M_{i+1}\}$  and  $\mathcal{B} \cap M_i$  is a subset of the Skolem hull of  $\{a_\eta \mid \eta \in J \cap M_{i+1}\}$ ,
- (v)  $\bigcup\{rk(\eta) \mid \eta \in I \cap M_i\} \cup \bigcup\{rk(\eta) \mid \eta \in J \cap M_i\} \in M_{i+1}$ .

By Theorem 16 (vi), the round can be played so that  $\forall$  loses. Let  $\alpha_i$ ,  $i < \omega$ , be the choices  $\exists$  made and  $\gamma$  such that  $((M_i)_{i < \omega}, (\alpha_i)_{i < \omega}) = (\overline{M}^\gamma, \eta^\gamma)$ . Finally, let  $\alpha = \bigcup_{i < \omega} \alpha_i \in U$ .

Now it is easy to see that  $\gamma \in W^\alpha$ , in fact  $\gamma \in W_I^\alpha$  or  $\gamma \in W_J^\alpha$  (otherwise we have demonstrated that  $\sigma$  is not a winning strategy). In the first case, there is a sequence  $a \in \mathcal{A}$  such that for all  $i < \omega$ ,  $a \not\prec_{C_i^\gamma} a_{\xi_{i+1}^\gamma}$  but in  $\mathcal{B}$  there is no sequence  $b$  such that for all  $i < \omega$ ,  $b \not\prec_{g^\gamma(C_i^\gamma)} g^\gamma(\xi_{i+1}^\gamma)$ , a contradiction. In the latter case, there is a sequence  $b \in \mathcal{B}$  such that for all  $i < \omega$ ,  $b \not\prec_{g^\gamma(C_i^\gamma)} g^\gamma(\xi_{i+1}^\gamma)$  but by (the construction,) Lemma 2.3 (c) and Theorem 16 (v), there is no sequence  $a \in \mathcal{A}$  such that for all  $i < \omega$ ,  $a \not\prec_{C_i^\gamma} a_{\xi_{i+1}^\gamma}$ , a contradiction.  $\square$

Now Lemmas 2.6 and 2.7 imply Theorem 2 (iii).

### 3.2 The unstable case

We will prove Theorem 2, case (i), again in a series of lemmas. We assume  $\omega_3^\omega < 2^{\omega_3}$ . Let  $T$  be a countable complete unstable first order theory. Let  $L$  be the signature of  $T$ .

**Theorem 19** ([11]) *Assume  $T$  is a countable unstable theory in the signature  $L$ . There are a countable signature  $L_1 \supseteq L$ , a complete Skolem theory  $T_1 \supseteq T$  in the signature  $L_1$ , a first-order  $L$ -formula  $\phi(x, y)$  and  $\Phi$  proper for  $(\omega, T_1)$  (see [Sh1] Definition VII 2.6) such that for every linear order  $I$  there is an Ehrenfeucht-Mostowski model  $EM^1(I, \Phi)$  of  $T_1$  with a skeleton  $\{a_\eta \mid \eta \in I\}$  such that*

$$EM^1(I, \Phi) \models \phi(a_\eta, a_\xi) \text{ iff } I \models \eta < \xi.$$

We write  $EM(I, \Phi)$  for  $EM^1(I, \Phi) \upharpoonright L$ . Notice that by using the terminology from [12, Definition III 3.1],  $\{a_\eta \mid \eta \in I\}$  is weakly  $(\omega, \phi)$ -skeleton-like in  $EM(I, \Phi)$ .

In order to use Theorem 19, linear orders are needed. If  $A$  is a linear ordering,  $x \in A$  and  $B \subseteq A$ , then by  $x < B$  we mean that for every  $y \in B$ ,  $x < y$ ,  $x > B$  and  $C > B$ ,  $C \subseteq A$  are defined similarly. By  $A^*$  we mean the inverse of  $A$ . Again let  $S, U$  and  $C_\alpha$ ,  $\alpha \in S$ , be as in [8, Lemma 16], i.e. Theorem 15 above, with the exception that  $0 \in S$  and for all  $\alpha \in S - \{0\}$ ,  $0 \in C_\alpha$ . By induction on  $i < \omega_3$ , we will define linear orders  $A_\alpha^i$  and  $B_\alpha^i$ ,  $\alpha < \omega_3$ , and for  $i \in S$ , isomorphisms

$$G_i : \Sigma_{\beta < i+2} A_\beta^i \rightarrow \Sigma_{\beta < i+2} B_\beta^i.$$

We write  $A^i(\beta, \alpha)$  for  $\Sigma_{\beta \leq \gamma < \alpha} A_\gamma^i$  and similarly  $B^i(\beta, \alpha)$ . We will do the construction so that the following holds:

- (1)  $A_\alpha^0 \cong \omega^*$  for all  $\alpha < \omega_3$  and if  $\alpha \notin U$ , then  $B_\alpha^0 \cong \omega^*$  and otherwise  $B_\alpha^0 \cong (\omega_1)^*$ ,
- (2) If  $i < j$ , then  $A_\alpha^i \subseteq A_\alpha^j$  and  $B_\alpha^i \subseteq B_\alpha^j$  and otherwise the sets are distinct and if  $j \in C_i$ , then  $G_j \subseteq G_i$ ,
- (3) if  $cf(\alpha) = \omega$ , then  $A_\alpha^0$  is coinital in  $A_\alpha^i$  and similarly for  $B$ .

We will do this by induction on  $i$ . However, in order to be able to show that (3) holds in each step, we need additional machinery.

Let  $C \in \{A, B\}$ . We say that  $(I, J)$  is a  $(C, i, \beta)$ -cut if  $I$  is an initial segment of  $C_\beta^i$  and  $J = C_\beta^i - I$ . We say that the cut is basic if  $I = \emptyset$ . We define a notion of forbidden cut by induction on  $i$  as follows (we should talk about  $i$ -forbidden cuts, but  $i$  is always clear from the context):

- (a) for all limit  $\beta$ , the basic  $(C, 0, \beta)$ -cut is forbidden,
- (b) if  $(I, J)$  is a  $(C, i, \beta)$ -cut,  $j < i$  and  $(C_\beta^j \cap I, C_\beta^j \cap J)$  is forbidden, then  $(I, J)$  is forbidden,
- (c) if  $(I, J)$  is a forbidden  $(A, i, \beta)$ -cut,  $I^* = I \cup \bigcup_{\gamma < \beta} A_\gamma^i$  and  $G_i(I^*)$  is not bounded by any  $x \in \bigcup_{\gamma < \delta} B_\gamma^i$  but some  $y \in B_\delta^i$  bounds it, then  $(G_i(I^*) \cap B_\delta^i, B_\delta^i - G_i(I^*))$  is forbidden and similarly for  $A$  and  $B$  reversed (and  $G_i$  replaced by  $(G_i)^{-1}$ ).

Now we can state the additional properties we want our construction have. Let  $E \in \{A, B\}$ ,  $i, \beta < \omega_3$  and  $(I, J)$  be a  $(E, i, \beta)$ -cut.

- (4) If  $(I, J)$  is forbidden, then there is no  $j < \omega_3$  and  $x \in E_\beta^j$  such that  $I < x < J$ .
- (5) Assume  $(I, J)$  is forbidden and  $j \in S$  is such that  $j < i$  and either  $E_\beta^j \cap I$  is cofinal in  $I$  or  $E_\beta^j \cap J$  is coinital in  $J$  (we say that  $\emptyset$  is both cofinal and coinital in  $\emptyset$ ). Then  $(E_\beta^j \cap I, E_\beta^j \cap J)$  is forbidden.
- (6) If  $\beta$  is successor, then  $E_\beta^0$  is coinital in  $E_\beta^i$ .

**Lemma 20** *Let  $E \in \{A, B\}$ .*

- (i) *For all  $i, \beta < \omega_3$ , if (5) holds upto the stage  $i$ , then  $(E_\beta^i, \emptyset)$  is not forbidden and neither is  $(\emptyset, E_\beta^i)$ , if  $\beta$  is successor.*
- (ii) *For limit  $\beta$ , every basic  $(E, i, \beta)$ -cut is forbidden.*
- (iii) *The property (4) implies the property (3).*
- (iv) *If  $i + 1 < \beta$  and  $(I, J)$  is a forbidden  $(E, i, \beta)$ -cut, then it is basic (and  $\beta$  is limit).*

**Proof.** Immediate.  $\square$

Now we are ready to do the construction: For  $i = 0$ , the linear orders are defined by (1) and we let  $G_0$  be the only possible one. Clearly (1)-(6) hold. If  $i \notin S$  or  $\sup C_i = i$ , then we let  $A_\alpha^i = \cup_{j < i} A_\alpha^j$ ,  $B_\alpha^i = \cup_{j < i} B_\alpha^j$  and if  $i \in S$  (and  $\sup C_i = i$ ), then  $G_i = G \cup \bigcup_{j \in C_i} G_j$ , where  $G$  is the obvious isomorphism from  $A^i(i, i+2)$  to  $B^i(i, i+2)$  (both are isomorphic to  $\omega^* + \omega^*$ ). Now (1), (2), (4) and (6) hold trivially. By Lemma 2.9 (iii), (3) holds. For (5), assume that  $C \in \{A, B\}$  and  $(I, J)$  is a forbidden  $(C, i, \beta)$ -cut. Now the reason why  $(I, J)$  is forbidden is (b) in the definition of forbidden cut (if  $i \notin S$ , then this is trivial and otherwise by the definition of  $G_i$ , (c) does not give forbidden cuts not forbidden by (b)). But then (5) follows immediately from the induction assumption.

We are left with the case  $i \in S$  and  $j = \sup C_i < i$ . Notice that now  $j \in C_i$ . Let  $\alpha < j + 2$  and  $A \neq \emptyset$  be an initial segment of  $A_\alpha^j$ . Let  $A^+ = A \cup \bigcup_{\gamma < \alpha} A_\gamma^j$ . Then there is the least  $\beta < j + 2$  such that  $B^+ = G_j(A^+) \cap (\cup_{\gamma \leq \beta} B_\gamma^j) = G_j(A^+)$ . Let  $A' = (A_\alpha^j \cup A_{\alpha+1}^j) - A$ ,  $B = G_j(A) \cap B_\beta^j$  and  $B' = (B_\beta^j \cup B_{\beta+1}^j) - B$ . Assume that at least one of  $C' = \{x \in \cup_{k < i} (A_\alpha^k \cup A_{\alpha+1}^k) \mid A < x < A'\}$  and  $D' = \{x \in \cup_{k < i} (B_\beta^k \cup B_{\beta+1}^k) \mid B < x < B'\}$  is non-empty. Then by the induction assumption,  $B \neq \emptyset$ . Let  $C$  be a copy of  $C'$  and  $D$  a copy of  $D'$ . Then we define  $A_\alpha^i$  so that it contains  $\cup_{k < i} A_\alpha^k$  and in each cut like above we add  $D$  so that  $A < C' < D < A'$  and  $B_\beta^i$  is defined similarly but now  $B < C < D' < B'$  (this is possible by (6) in the induction assumption). Then, by (4) in the induction assumption, we can find an isomorphism  $G'_i : \cup_{\alpha < j+2} A_\alpha^i \rightarrow \cup_{\alpha < j+2} B_\alpha^i$ . Notice that by (5) in the induction assumption, for all  $\delta < i$ , the  $(A, \delta, \alpha)$ -cut  $(A_\alpha^\delta - A(\delta), A(\delta))$  and  $(B, \delta, \beta)$ -cut  $(B(\delta), B_\beta^\delta - B(\delta))$  are not forbidden, where  $A(\delta) = \{x \in A_\alpha^\delta \mid x > C'\}$  and  $B(\delta) = \{x \in B_\beta^\delta \mid x < D'\}$ . So we have not violated the property (4).

For all  $\alpha > j + 1$ , we let  $A_\alpha^i = \cup_{k < i} A_\alpha^k$  and  $B_\alpha^i$  is defined similarly. However we will still make changes to  $B_{j+1}^i$  and  $A_{i+1}^i$ ! Let  $A$  be a copy of  $B^i(j+3, i+2)$  and  $B$  be a copy of  $A^i(j+2, i+1)$ . Furthermore, extend  $A_{i+1}^i$  so that there is an isomorphism  $g : A_{i+1}^i \rightarrow B_{j+2}^i$  such that  $g(A_{i+1}^0) = B_{j+2}^0$  (this is not a problem since  $A_{i+1}^0 = \cup_{k < i} A_{i+1}^k \cong \omega^* \cong B_{j+2}^0$  and by Lemma 2.9 (iv), the sets  $A_{i+1}^k$ ,  $k < i$ , do not contain forbidden  $(A, k, i+1)$ -cuts; so we do not violate (4)). Then we add  $A$  to (the extended)  $A_{i+1}^i$  as an end segment and  $B$  to  $B_{j+1}^i$  as an end segment. By Lemma 2.9 (i), this does not violated (4). Now it is easy to extend  $G'_i$  to  $G_i$  so that  $G_i(A^i(j+2, i+1)) = B$ ,  $G_i(A_{i+1}^i - A) = B_{j+2}^i$  and  $G_i(A) = B^i(j+3, i+2)$ .

Now (1), (2) and (6) hold trivially, (4) is already shown to hold and by Lemma 2.9 (iii), (4) implies (3). So we are left to show that

**Lemma 21** (5) holds.

**Proof.** Assume  $(I, J)$  is a forbidden  $(E, i, \beta)$ -cut,  $E \in \{A, B\}$ , and  $\delta \in S$  is such that  $\delta < i$  and  $E_\beta^\delta \cap J$  is coinital in  $J$ , the other case is similar. If  $\beta \geq j + 1$  and both  $J \cap (\cup_{k < i} A_{j+1}^k)$  and  $J \cap (\cup_{k < i} B_{j+1}^k)$  are empty, then the claim follows easily from Lemma 2.9 and the induction assumption. So we assume that this is not the case. If  $(I, J)$  is forbidden because of (b) in the definition of forbidden cut, the claim follows from the induction assumption. So we assume that  $E = B$  and there is a forbidden  $(A, i, \gamma)$ -cut  $(C, D)$  such that  $(I, J)$  is forbidden by (c) applied to  $(C, D)$  (the case  $A$  and  $B$  reversed is symmetric). Since  $(I, J)$  is not forbidden by (b) in the definition of forbidden cut,  $(C, D)$  must be forbidden because of it, i.e. for some  $\alpha < i$ ,  $(A_\gamma^\alpha \cap C, A_\gamma^\alpha \cap D)$  is a forbidden  $(A, \alpha, \gamma)$ -cut.

If

( $\star$ ) For no  $y \in A_\gamma^\alpha \cap D$ ,  $y < A_\gamma^j \cap D$ ,

then by the induction assumption,  $(B_\beta^j \cap I, B_\beta^j \cap J)$  is a forbidden  $(B, j, \beta)$ -cut and the claim follows from the definition of forbidden cut if  $\delta \geq j$  and from (5) in the induction assumption if  $\delta < j$ . So we assume that ( $\star$ ) fails. Let  $y$  be the bound. Then  $\emptyset \neq D' = \{z \in A_\gamma^i \cap D \mid z \leq y\} \subseteq A_\gamma^i - \text{dom}(G_j)$ . So by the construction,  $G_i(D') \subseteq J - B_\beta^\delta$  and for all  $x \in B_\beta^\delta$ , either  $x < G_i(D')$  or  $x > G_i(D')$ . By the choice of the cut  $(C, D)$ , there can not be  $x \in J \cap B_\beta^\delta$  such that  $x < G_i(D')$ . But then  $G_i(D') < J \cap B_\beta^\delta$ , which contradicts the assumption that  $J \cap B_\beta^\delta$  is coinital in  $J$ .  $\square$

Let  $A = \sum_{\alpha < \omega_3} \cup_{i < \omega_3} A_\alpha^i$  and  $B = \sum_{\alpha < \omega_3} \cup_{i < \omega_3} B_\alpha^i$ . Notice that by (1) and (3),  $\text{inv}_\omega^1(A)$  differs from  $\text{inv}_\omega^1(B)$  in a stationary set which consists of ordinals of cofinality  $\omega$  (for the definition of  $\text{inv}_\omega^n$ , see [12, Definition III 3.4]). Let  $S_\alpha \subseteq S_0^3$ ,  $i < 2^{\omega_3}$ , be stationary sets such that for  $\alpha < \beta < 2^{\omega_3}$ ,  $S_\alpha \triangle S_\beta$  is stationary and define  $\Psi_\alpha = \sum_{\alpha < \omega_3} \tau_\alpha$ , where  $\tau_\alpha = A^*$  if  $\alpha \notin S_\alpha$  and otherwise  $\tau_\alpha = B^*$ . Notice that for  $\alpha \neq \beta$ ,  $\text{inv}_\omega^2(\Psi_\alpha)$  differs from  $\text{inv}_\omega^2(\Psi_\beta)$  in a stationary set which consists of ordinals of cofinality  $\omega$ .

Finally, let  $\mathcal{A}_\alpha = EM((\Psi_\alpha)^* \cdot \omega_1, \Phi)$ .

**Lemma 22** For all  $\alpha, \beta < 2^{\omega_3}$ ,  $A$  does not have a winning strategy for  $EF_{\omega_1}^{\omega_3}(\mathcal{A}_\alpha, \mathcal{A}_\beta)$ .

**Proof.** For this, it is enough to show that  $A$  does not have a winning strategy for  $EF_{\omega_1}^{\omega_3}((\Psi_\alpha)^* \cdot \omega_1, (\Psi_\beta)^* \cdot \omega_1)$ , which follows easily from (2) in the construction of  $A$  and  $B$  and Theorem 15 (see e.g. [8, Claim 3 in the proof of Theorem 17]).  $\square$

**Lemma 23** *There are  $\alpha < \beta < 2^{\omega_3}$  such that  $E$  does not have a winning strategy for  $EF_{\omega_1}^2(\mathcal{A}_\alpha, \mathcal{A}_\beta)$ .*

**Proof.** By using the usual forcing notion, we collapse  $\omega_3$  to an ordinal of power  $\omega_1$ . Since this forcing notion does not kill those stationary subsets of  $\omega_3$  which consist of ordinals of cofinality  $\omega$  and cofinalities  $\leq \omega_1$  are preserved, in the extension,  $inv_\omega^2(\Psi_\alpha) \neq inv_\omega^2(\Psi_\beta)$  for all  $\alpha \neq \beta$ . Clearly, the skeletons of the models  $\mathcal{A}_\alpha$ , remain weakly  $(\omega, \phi)$ -skeleton-like in  $\mathcal{A}_\alpha$ . So by (the proof of) [12, Lemma III 3.15 (1)],  $inv_\omega^2(\Psi_\alpha) \in INV_\omega^2(\mathcal{A}_\alpha, \phi)$  in the extension (for the definition of  $INV_\omega^n$ , see [12, Definition III 3.11] and notice that  $\mathcal{A} \cong \mathcal{B}$  implies  $INV_\omega^2(\mathcal{A}, \phi) = INV_\omega^2(\mathcal{B}, \phi)$ ). Also by [Sh2] Lemma III 3.13 (1),  $|INV_\omega^2(\mathcal{A}_\alpha, \phi)| = \omega_1$ . Since  $\omega_3^\omega < 2^{\omega_3}$  in the ground model, in the generic extension,  $(2^{\omega_3})^V$  is a cardinal  $> \omega_1$ . So there are  $\alpha < \beta < (2^{\omega_3})^V$  such that  $\mathcal{A}_\alpha \not\cong \mathcal{A}_\beta$  in the extension. Since countable subsets are not added,  $E$  does not have a winning strategy for  $EF_{\omega_1}^2(\mathcal{A}_\alpha, \mathcal{A}_\beta)$  (in the ground model).  $\square$

Now Lemmas 2.11 and 2.12 imply Theorem 2 (i).  $\square$

Before proving the theorem, we make some remarks which follow from the proof.

**Remark 24** In many cases in Theorem 2, the assumption on  $2^\omega$  can be removed. For example, this is true of linear orders. An easy proof for this is given in [2], alternatively this follows immediately from the proof of Theorem 2 (i) by checking where the assumption  $2^\omega < 2^{\omega_3}$  was needed. Another case where the assumption on  $2^\omega$  can be removed is the case that  $\theta = \omega_3$  in the stable unsuperstable case. This follows from the proof of Theorem 2 (iii) by noticing that the black box can now be replaced by an argument from [3]. Another remark is that in Theorem 2 (i) and (ii),  $\omega_3$  can be replaced by any cardinal  $\kappa \geq \omega_3$  such that  $\kappa$  is a successor of a regular cardinal and  $2^\kappa > \kappa^\omega$ . Finally, in Theorem 2 (iii),  $\omega_3$  can be replaced by any cardinal  $\kappa \geq \omega_3$  such that  $\kappa$  is a successor of a regular cardinal and  $\kappa^\omega = \kappa$ .

## References

- [1] Matthew Foreman. Games played on Boolean algebras. *J. Symbolic Logic* 48 (1983), no. 3, 714–723.
- [2] Taneli Huuskonen, Comparing notions of similarity for uncountable models, *Journal of Symbolic Logic*, 60, 1995, 1153–1167,
- [3] Tapani Hyttinen and Saharon Shelah, On the number of elementary submodels of an unsuperstable homogeneous structure, *Mathematical Logic Quarterly* (44) 1998, 354–358.
- [4] Tapani Hyttinen and Heikki Tuuri, Constructing strongly equivalent nonisomorphic models for unstable theories, *Annals of Pure and Applied Logic* (52) 1991.
- [5] T. Jech, *Set Theory*, Academic Press, 1978.
- [6] T. Jech, M. Magidor, W. Mitchell and K. Prikry, Precipitous ideals, *Journal of Symbolic Logic* 45 (1980), 1–8.
- [7] Menachem Magidor, Reflecting stationary sets, *Journal of Symbolic Logic*, 47, 1982, 755–771
- [8] Alan H. Mekler, Saharon Shelah, and J. Väänänen. The Ehrenfeucht-Fraïssé-game of length  $\omega_1$ . *Transactions of the American Mathematical Society*, **339**:567–580, 1993.
- [9] Taneli Huuskonen, Tapani Hyttinen and Mika Rautila. On potential isomorphism and non-structure, to appear.
- [10] Saharon Shelah. Reflecting stationary sets and successors of singular cardinals. *Archive for Mathematical Logic*, **31**:25–53, 1991.
- [11] Shelah, Saharon, *Classification theory and the number of nonisomorphic models*, Second edition, North-Holland Publishing Co., Amsterdam, xxxiv+705 pp, *Studies in Logic and the Foundations of Mathematics*, Vol. 92, 1990
- [12] Shelah, *Non-structure Theory*, to appear.

- [13] Shelah, Saharon and Stanley, Lee, A theorem and some consistency results in partition calculus, *Annals of Pure and Applied Logic*, 36, 1987, 119–152.
- [14] Shelah, Saharon and Stanley, Lee, More consistency results in partition calculus, *Israel Journal of Mathematics*, 81, 1993, 97–110.