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REPORT No. 19, 2001/2002

ISSN 1103-467X
ISRN IML-R- -19-01/02- -SE

INSTITUT MITTAG-LEFFLER
THE ROYAL SWEDISH ACADEMY OF SCIENCES

SUPER-BROWNIAN MOTION WITH REFLECTING HISTORICAL PATHS. II CONVERGENCE OF APPROXIMATIONS

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Abstract

We prove that the sequence of finite reflecting branching Brownian motion forests defined by Burdzy and Le Gall ([1]) converges in probability to the "super-Brownian motion with reflecting historical paths." This solves an open problem posed in [1], where only tightness was proved for the sequence of approximations. Several results on path behavior were proved in [1] for all subsequential limits—they obviously hold for the unique limit found in the present paper.

AMS 1991 *subject classifications.* Primary 60H15. Secondary 35R60.

Keywords and phrases. Super-Brownian motion, reflecting paths, Brownian snake, martingale problem.

Running head. Super-Brownian motion with reflection

Supported in part by NSF Grant DMS-0071486, Israel Science Foundation Grants 12/98 and 116/01 - 10.0, and the U.S.-Israel Binational Science Foundation (grant No. 2000065)

1 Introduction

The goal of this paper is to complete the main stage of a research project started by Burdzy and Le Gall in [1]. The authors of that paper set out to define and study a "super-Brownian motion with reflecting historical paths." They constructed a sequence of branching particle systems which they believed converged to a limit representing, in the intuitive sense, the process named above. However, the main result of [1] proves only tightness for the sequence of approximations. This will be remedied in this paper—we will prove the convergence. Moreover, we will construct a sequence of approximations which converges not only in the sense of distribution but also in probability. This will allow us to complete the definition of a "super-Brownian motion with reflecting historical paths" in the sense that our main result will identify a single probability distribution on an appropriate space. A rigorous statement of our main result is given as Theorem 2.1 in Section 2—it requires a fair amount of notation.

We would like to mention that some path properties have been already established for the super-Brownian motion with reflecting historical paths in [1]. The properties have been proved for every subsequential limit of the approximating sequence of branching particle systems, so those theorems obviously apply to the process constructed in this paper.

Our approach is closely related to and heavily dependent on techniques developed in [1], and that paper in turn uses many tools presented in [4], in particular the "Brownian snake" originally introduced by Le Gall. For an introduction to the theory of superprocesses see, for example, [3].

We will now describe our main ideas. Just as in [1], we start with a sequence of branching particle systems approximating the historical super-Brownian motion. The accuracy of the approximation is determined by a parameter $\epsilon > 0$. The sequence of approximations is consistent in the sense that ϵ-particle system may be obtained from a δ-particle system by pruning some branches, for any $\delta < \epsilon$. Then paths of every ϵ-particle system are relabelled so that the paths in the new system are reflecting. The problem with this approach is that the reflecting systems are no longer consistent, i.e., after relabelling, the distance (in an appropriate metric) between reflecting ϵ-particle system and reflecting δ-particle system may be large, at least for some ω. To tackle this problem, we will first consider two reflecting super-Brownian motions. In this model, particles of one system reflect with the particles of the other system, and since the model is non-historical, reflections between particles of the same system are irrelevant to the evolution of two reflecting super-Brownian motions. Intuitively speaking, the "number" of reflections is much smaller in this model than in the fully reflecting historical super-Brownian motion. This allows us to prove convergence of appropriate finite particle system approximations in this model. The result generalizes to any finite number of reflecting super-Brownian motions (again, in the non-historical setting reflections between the particles of the same particle system are irrelevant). Finally, the super-Brownian motion with reflecting historical paths is obtained as the limit of families of reflecting historical Brownian motions, with larger and larger numbers of particle systems in the families.

Section 2 contains the construction of a finite particle system and the statements of our main results. Section 3 is a review of some relevant facts on local times and excursions. Section 4 presents a construction of a pair ("coupling") of finite particle systems reflecting with each other. It also contains the proof of convergence of such approximations. The main result of this paper is proved in Section 5.

Acknowledgements. The first author would like to express his gratitude for the opportunity to visit Technion (Israel) and Institut Mittag-Leffler (Sweden) where this research was partially done. The second author is grateful to Jean-François Le Gall for several helpful discussions of the problem.

Notation We will adopt the following notation conventions.
Let E be a Polish space.
$M_F(E)$ (M_F) — finite measures on E (on $\mathbb{R}$).
$D_E[a,b)$ (D_E) — Skorohod space of cadlag E-valued paths on $[a,b)$ (on $[0,\infty)$).
$C_E[a,b)$ (C_E) — the space of continuous E-valued paths on $[a,b)$ (on $[0,\infty)$).
$C(E)$ (C) — the space of bounded, real-valued, continuous functions on E (on $\mathbb{R}_+$).
$C_u(E)$ — the space of bounded, real-valued, uniformly continuous functions on E.
$B(E)$ — the space of bounded Borel measurable real-valued functions on E.
$\langle \mu, f\rangle = \int f d\mu$ for any measure μ and function f.
We will try to use as much as possible of the notation from [1] to help the reader follow our arguments, as we will often refer to that paper.

2 Finite branching particle systems and the statement of main results

We will be brief in our presentation of super-Brownian motion, historical processes, finite particle systems, Brownian snake, etc. The reader is asked to consult [1] and [4] for more details.

First we are going to introduce the historical super-Brownian motion based on Le Gall's Brownian snake construction. Let $\mu \in M_F$ and β be twice the reflecting Brownian motion, i.e., $\{\beta_s, s\geq 0\} \stackrel{\text{dist}}{=} \{2|B_s|, s\geq 0\}$, stopped at time $\tau = \inf\{t : L^0_t = \langle \mu, 1\rangle\}$, where B_s is the standard Brownian motion with $B_0 = 0$ and $(L^x_s, x\geq 0, s\geq 0)$ denotes the jointly continuous family of local times of β normalized in such a way that, for every nonnegative Borel function ϕ on $\mathbb{R}_+$ and $t>0$,

$$\int_0^t \phi(\beta_s)\, ds = \int_{\mathbb{R}_+} \phi(x) L^x_t \, dx.$$

We will assume that μ is absolutely continuous with respect to the Lebesgue measure, to simplify the proofs. Let $\{W_s, s\geq 0\}$ be the Brownian snake driven by the process β and such that

$$\mu = \int_0^\tau dL^0_s\, \delta_{W_s(0)}\,. \tag{2.1}$$

Recall that for any fixed $s\geq 0$, $\{W_s(t), t\geq 0\}$ is a Brownian motion stopped at time β_s. For more details see Theorem IV.4 of Le Gall [4]. The historical super-Brownian motion is defined by

$$Y_t = \int_0^\tau dL^t_s\, \delta_{W_s}\,.$$

The corresponding M_F-valued process (super-Brownian motion) is defined by

$$X_t = \int_0^\tau dL^t_s \delta_{W_s(t)}\,.$$

Next we will define finite branching particle systems approximating the historical super-Brownian motion. Fix an arbitrary $\epsilon > 0$. For any $s\geq 0$, $t\geq 2\epsilon$, let $W^{t-2\epsilon}_s$ be the path of the Brownian snake with index s stopped at time $t-2\epsilon$. Let

$$H^\epsilon_t \equiv \int_0^\tau dL^t_s\, \delta_{W^{t-2\epsilon}_s}\,, \quad t\geq 2\epsilon.$$

In other words, H^ϵ_t is the family of trajectories of those particles which are alive at time t, but the paths in H^ϵ_t are stopped at time $t-2\epsilon$. The measure H^ϵ_t is purely atomic. Next we redefine the masses of these atoms—we give mass ϵ to each atom of H_t^ϵ for any $t \geq 2\epsilon$. The resulting measure-valued process will be denoted by $\{Y_t^\epsilon, t \geq 2\epsilon\}$. Note that Y^ϵ represents binary branching Brownian motions with branching rate ϵ^{-1}. The initial positions of the particles are distributed according to the Poisson measure on $\mathbb{R}$ with intensity $\epsilon^{-1}\mu$ (see Proposition 3.5 of [2]). The M_F-valued process corresponding to Y^ϵ is defined by

$$\langle X_t^\epsilon, \phi\rangle = \int \phi(y(t)) Y_t^\epsilon(dy), \quad \forall \phi \in C(\mathbb{R}).$$

It was explained in Section 2.2 of [1] how any finite branching particle system can be coded as a marked forest consisting of the set $\mathcal{T}^\epsilon$ of edges (i.e., particles) which is a subset of

$$\{1,2,\dots\} \times \bigcup_{n=0}^{\infty} \{1,2\}^n,$$

and the family $(l_u^\epsilon, u \in \mathcal{T}^\epsilon)$ of lengths of edges (i.e., lifetimes of particles). Let $(\mathcal{T}^\epsilon, (l_u^\epsilon, u \in \mathcal{T}^\epsilon))$ be a marked forest representing the genealogical structure of Y^ϵ and let $(\beta^\epsilon, s \in [0, \tau^\epsilon])$ be the random walk corresponding to this marked forest (see Section 2.2 of [1]). Note that the spatial positions of the particles are irrelevant for the construction of β^ϵ—what matters is β itself.

It easily follows from the definition of Y^ϵ and Theorem 3.10 of [2]) that as $\epsilon \to 0$,

$$Y_t^\epsilon \to Y_t, \quad \text{in } M_F(C), \quad P - \text{a.s.}, \ \forall t > 0,$$

and

$$Y_\cdot^\epsilon \to Y_\cdot, \quad \text{in } D_{M_F(C)},$$

in probability (see Remark 2 after Theorem 3.10 of [2]). Also

$$X_\cdot^\epsilon \to X_\cdot, \quad \text{in } D_{M_F},$$

in probability.

We will now discuss reflecting binary branching Brownian motions. The closed support of a measure ν on $\mathbb{R}$ will be denoted $\text{supp}(\nu)$, i.e., $\text{supp}(\nu)$ is the smallest closed set A such that $\nu(A^c) = 0$. Recall μ and τ from the beginning of this section. Fix arbitrary positive measures μ_1 and μ_2 which satisfy the following assumptions:

1. $\mu_1 + \mu_2 = \mu$.
2. The support of μ_1 lies to the left of the support of μ_2, that is, for any $x_i \in \text{supp}(\mu_i)$, $i = 1, 2$, we have $x_1 \leq x_2$.

Let

$$\tau_1 = \inf\{t : L_t^0 = \langle \mu_1, 1\rangle\},$$

$$Y_t^1 = \int_0^{\tau_1} dL_s^t \, \delta_{W_s}, \qquad Y_t^2 = \int_{\tau_1}^{\tau} dL_s^t \, \delta_{W_s}.$$

Then Y^1 and Y^2 are two historical super-Brownian motions starting at μ_1 and μ_2 respectively. We define their approximations $Y^{1,\epsilon}$ and $Y^{2,\epsilon}$ in the same way as Y^ϵ was defined for the process Y. Note that

$$\begin{aligned} Y &= Y^1 + Y^2, \\ Y^\epsilon &= Y^{1,\epsilon} + Y^{2,\epsilon}. \end{aligned}$$

The process $\{Y_t^{i,\epsilon}, t \geq 2\epsilon\}$ represents the historical branching particle system with the initial positions of particles distributed according to the Poisson random measure with intensity μ^i/ϵ, $i = 1, 2$. Arguing as in Section 3 of [1], we relabel the paths of Y^ϵ to obtain a reflecting branching particle system $\widetilde{Y}^\epsilon$. If $\widetilde{w}(t)$ and $\widetilde{w}'(t)$ are atoms of $\widetilde{Y}^\epsilon$ and ζ is the minimum of lifetimes of $\widetilde{w}(t)$ and $\widetilde{w}'(t)$ then we either have $\widetilde{w}(t) \geq \widetilde{w}'(t)$ for all $t \in [0, \zeta]$ or $\widetilde{w}(t) \leq \widetilde{w}'(t)$ for all $t \in [0, \zeta]$. We will denote by $\widetilde{Y}^{1,\epsilon}$ the subsystem of $\widetilde{Y}^\epsilon$ consisting of those trees in $\widetilde{Y}^\epsilon$ which start at time 0 from the same points as $Y^{1,\epsilon}$, and similarly we define $\widetilde{Y}^{2,\epsilon}$. Hence,

$$\widetilde{Y}^\epsilon \equiv \widetilde{Y}^{1,\epsilon} + \widetilde{Y}^{2,\epsilon}.$$

It follows from the definition that the paths inside of each of $\widetilde{Y}^{i,\epsilon}, i = 1, 2$, are reflecting. The corresponding M_F-valued processes are defined by

$$\langle \widetilde{X}_t^{i,\epsilon}, \phi \rangle = \int \phi(y(t)) \widetilde{Y}_t^{i,\epsilon}(dy), \quad \forall \phi \in C(\mathbb{R}), \quad i = 1, 2.$$

The aim of this paper is to prove the following result.

Theorem 2.1 *There exists a process* $\widetilde{Y} \in C_{M_F(C)}[0, \infty)$ *such that*

$$\widetilde{Y}^\epsilon \to \widetilde{Y}, \text{ as } \epsilon \downarrow 0, \text{ in } D_{M_F(C)}[0, \infty),$$

in probability.

The crucial step in the proof will be the following theorem.

Theorem 2.2 *There exists a process* $\left(\widetilde{X}^1, \widetilde{X}^2\right) \in C_{M_F \times M_F}[0, \infty)$ *such that*

$$\left(\widetilde{X}^{1,\epsilon}, \widetilde{X}^{2,\epsilon}\right) \to \left(\widetilde{X}^1, \widetilde{X}^2\right), \text{ as } \epsilon \downarrow 0, \text{ in } D_{M_F \times M_F}[0, \infty),$$

in probability.

3 Review of local times and excursions

Fix an arbitrary $a > 0$. Recall that β has the distribution of the twice of reflecting Brownian motion, $\beta_0 = 0$, and L_t^x is the family of local times of β. Here we will assume that β is stopped at time

$$\tau = \tau_a = \inf\{t : L_t^0 = a\}.$$

For any $s \geq 0$ and $\nu > 0$ let

$$\eta_1^\nu(s) = \begin{cases} \sup\{t : \int_t^\tau d_u L_u^s \geq \nu\}, & L_\tau^s \geq \nu, \\ \infty, & \text{otherwise,} \end{cases}$$

$$\eta_2^\nu(s) = \begin{cases} \inf\{t : \int_0^t d_u L_u^s \geq \nu\}, & L_\tau^s \geq \nu, \\ 0, & \text{otherwise.} \end{cases}$$

For any $s \geq 0$, denote by $(a_i^s, b_i^s), i = 1, 2, \ldots$ the excursion intervals of β above level s, where the ordering of this countably infinite set of intervals is arbitrary, for example, it may be the ordering according to the decreasing length. For any i denote by e_i^s the corresponding excursion

$$e_i^s(u) = \beta_{(a_i^s + u) \wedge b_i^s} - s, \quad u \geq 0.$$

For $t > s \geq 0$ let $L_i^{s,t}(\beta)$ be the local time of excursion e_i^s at level t. Note that

$$L_\tau^t(\beta) = \sum_i L_i^{s,t}(\beta).$$

Let $I_s^{\nu,1}$ be the set of indices of excursions originating at level s in the interval $[\eta_1^\nu(s), \infty)$. More precisely

$$I_s^{\nu,1} = \{i : a_i^s \geq \eta_1^\nu(s)\}.$$

Analogously

$$I_s^{\nu,2} = \{i : b_i^s \leq \eta_2^\nu(s)\}.$$

Denote

$$L^{s,t,j}(\beta,\nu) = \sum_{i \in I_s^{\nu,j}} L_i^{s,t}(\beta), \quad j = 1,2,$$

$$L^{s,j}(\beta,\nu) = \sup_{t \geq s} L^{s,t,j}(\beta,\nu), \quad j = 1,2,$$

$$L^j(\beta,\nu) = \sup_{s \geq 0} L^{s,j}(\beta,\nu), \quad j = 1,2.$$

In other words, $L^1(\beta,\nu)$ is the maximum total local time at any level of any excursion originating at any $[\eta_1^\nu(s),\infty)$, $s \geq 0$, and $L^2(\beta,\nu)$ is the corresponding quantity for excursions originating at $[0, \eta_2^\nu(s))$, $s \geq 0$.

In the following, P^a will refer to the law of β starting at 0 and stopped at τ_a. If there is no ambiguity, we will suppress the dependence on β in the notation.

Lemma 3.1 *For any $a > 0$ and arbitrary small $p, \overline{\alpha} > 0$ there exists $\nu^* = \nu^*(p,\overline{\alpha}) > 0$ such that*

$$\sup_{i=1,2} \sup_{b \leq a} P^b\left(L^i(\nu) \geq \overline{\alpha}\right) \leq p/8, \quad \forall \nu \leq \nu^*.$$

Proof The quantity $P^b\left(L^i(\nu) \geq \overline{\alpha}\right)$ is a non-decreasing function of b so it is enough to prove that for some $\nu^* > 0$ and all $\nu \leq \nu^*$ we have $P^a\left(L^i(\nu) \geq \overline{\alpha}\right) \leq p/8$, for $i = 1,2$. We will discuss only the case $i = 1$. The function $\nu \to L^1(\nu)$ is non-decreasing so it will suffice to show that $\lim_{\nu \to 0} L^1(\nu) = 0$, P^a-a.s. Suppose that $\lim_{\nu \to 0} L^1(\nu) \neq 0$ with positive probability. We will show that this assumption leads to a contradiction. Fix an ω such that $\lim_{k\to\infty} L^1(1/k) = c > 0$ and find a sequence of levels $t_k > s_k$ such that $L^{s_k,t_k,1}(1/k) \geq c$ for all k. For each k we choose an excursion $e_k^{s_k}$ from level s_k, with endpoints $a_k^{s_k}$ and $b_k^{s_k}$ such that $a_k^{s_k} \geq \eta_1^{1/k}(s_k)$ and $\sup_{u\geq 0} e_k^{s_k}(u) \geq t_k - s_k$. By compactness, we may assume that $s_k \to s_\infty$, $t_k \to t_\infty$, $a_k^{s_k} \to a_\infty$ and $b_k^{s_k} \to b_\infty$. Recall that L_t^x is jointly continuous in t and x. If $t_\infty = s_\infty$ then it is easy to see that $x \to L_{b_\infty}^x$ has a discontinuity at $x = s_\infty$, which can happen only for a set of ω's of probability 0. Suppose that $t_\infty > s_\infty$. Let $t' = (t_\infty + s_\infty)/2$ and let s' be the supremum of $s < b_\infty$ with $\beta_s = t'$. Let $\widehat{L}_t^x$ be the jointly continuous family of local times for the process $\{\beta_s, s \geq s'\}$. For large k we have $\widehat{L}_{b_k^{s_k}}^{s_k} = 0$, so $\widehat{L}_{b_\infty}^{s_\infty} = 0$. Since $a_k^{s_k} \geq \eta_1^{1/k}(s_k)$, it follows that

$$\widehat{L}_\infty^{s_k} - \widehat{L}_{b_k^{s_k}}^{s_k} \leq \widehat{L}_\infty^{s_k} - \widehat{L}_{a_k^{s_k}}^{s_k} \leq 1/k,$$

and this in turn implies that $\widehat{L}_\infty^{s_\infty} - \widehat{L}_{b_\infty}^{s_\infty} = 0$. Hence, $\widehat{L}_\infty^{s_\infty} = 0$. This can happen only for a negligible set of ω's because with probability 1, for every rational $s' < \tau_a$, the family of local times

$\widehat{L}_t^x$ corresponding to $\{\beta_s, s \geq s'\}$ has the property $\inf_{x\in[0,\beta_{s'}]} \widehat{L}_\infty^x > 0$, by the Ray-Knight theorem. ■

Recall from Section 2 the definition of β^ϵ, a continuous time random walk corresponding to β. Let

$$L_u^{\epsilon,t} = L_u^{\epsilon,t}(\beta^\epsilon) = \epsilon \mathrm{Card}\{r \in [0,u) : \beta_r^\epsilon = t \text{ and } \beta_v^\epsilon > t, \text{ for } v \in (r, r+\delta], \text{ for some } \delta > 0\}.$$

In other words, $\epsilon^{-1} L_s^{\epsilon,t}$ is the number of upcrossings of β^ϵ above level t before time s. Then, for any $s \geq 0$ and $\nu > 0$ let

$$\begin{aligned} \eta_1^{\epsilon,\nu}(s) &= \begin{cases} \sup\{t : \int_t^{\mathcal{T}} d_u L_u^{\epsilon,s} \geq \nu\}, & L_{\mathcal{T}^\epsilon}^{\epsilon,s} \geq \nu, \\ \infty, & \text{otherwise,} \end{cases} \\ \eta_2^{\epsilon,\nu}(s) &= \begin{cases} \inf\{t : \int_0^t d_u L_u^{\epsilon,s} \geq \nu\}, & L_{\mathcal{T}^\epsilon}^{\epsilon,s} \geq \nu, \\ 0, & \text{otherwise.} \end{cases} \end{aligned}$$

For any $s \geq 0$, denote by $(a_i^{s,\epsilon}, b_i^{s,\epsilon}), i = 1, 2, \ldots$ the excursion intervals of β^ϵ above level s, and denote by $e_i^{s,\epsilon}$ the corresponding excursions. For $t > s \geq 0$ let $L_i^{s,t,\epsilon}(\beta^\epsilon)$ be the rescaled number of upcrossings of excursion $e_i^{s,\epsilon}$ above level t. Let

$$I_s^{\nu,1,\epsilon} = \{i : a_i^{s,\epsilon} \geq \eta_1^{\epsilon,\nu}(s)\}.$$

Analogously

$$I_s^{\nu,2,\epsilon} = \{i : b_i^{s,\epsilon} \leq \eta_2^{\epsilon,\nu}(s)\}.$$

Denote

$$L^{j,\epsilon}(\beta^\epsilon, \nu) = \sup_{t \geq s \geq 0} \sum_{i \in I_s^{\nu,j,\epsilon}} L_i^{s,t,\epsilon}(\beta^\epsilon, \nu), \quad j = 1, 2.$$

Lemma 3.2 *Let β^ϵ, β be as above. Let $p, \overline{\alpha}, \nu^*$ be as in the previous lemma. Then there exists ϵ_1 sufficiently small such that for all $\epsilon \leq \epsilon_1$ and $\nu \leq \nu^*$,*

$$\sup_{i=1,2} \sup_{b \leq a} P^b\left(L^{i,\epsilon}(\nu) \geq 2\overline{\alpha}\right) \leq p/4.$$

Proof The lemma follows from our Lemma 3.1 and Lemma 2.1 of [1]. ■

Remark 3.3 *It is legitimate to use notation P^b in the above lemma since β^ϵ is a function of β.*

Let X^ϵ be the M_F-valued process constructed in Section 2.

Lemma 3.4 *Fix ν, p arbitrary small.*

(i) *There exist $\alpha = \alpha(\nu, p), \epsilon_2 = \epsilon_2(\nu, p) > 0$, such that*

$$P\left(\sup_{t \geq 2\epsilon, x \in \mathbb{R}} X_t^\epsilon(x, x+\alpha) \leq \nu/2\right) \geq 1 - p/4, \quad \forall \epsilon \leq \epsilon_2.$$

(ii) *For any $\overline{\alpha}$ arbitrary small, there exists $\epsilon_3 = \epsilon_3(\overline{\alpha}, p)$ such that*

$$P\left(\sup_{t \geq 2\max(\epsilon', \epsilon'')} \sup_{x \in \mathbb{R}, r \geq 0} \left| X_t^{\epsilon'}(B(x,r)) - X_t^{\epsilon''}(B(x,r)) \right| \geq \overline{\alpha}\right) \leq p/2, \quad \forall \epsilon', \epsilon'' \leq \epsilon_3.$$

Proof Recall from Section 2 that $X^\epsilon \to X$, in D_{M_F}, in probability, as $\epsilon \downarrow 0$. It is well known that $X \in C_{M_F}$, and $X_t(dx)$ is absolutely continuous with respect to the Lebesgue measure for every t. The lemma follows easily from these facts.

■

Lemma 3.5

(a) $$\lim_{\delta\downarrow 0} \sup_{s,t} |W_s(t+\delta) - W_s(t)| = 0, \quad P - \text{a.s.},$$

(b) $$\lim_{\delta\downarrow 0} \left(\limsup_{\epsilon\downarrow 0} P\left(\sup_{s\geq 2\epsilon} \sup_{|t-t'|\leq\delta} \sup_{\widetilde{y}\in\text{supp}(\widetilde{Y}^\epsilon_s)} \left|\widetilde{y}(t) - \widetilde{y}(t')\right| > \eta \right)\right) = 0, \quad \forall \eta > 0. \ P - \text{a.s.}$$

Proof **(a)** is a well known fact, see, for example, (2.7) of [1]. For **(b)** see Theorem 4.1 of [1].

■

The last lemma easily implies the following.

Corollary 3.6 *For any $\alpha, p > 0$, there exists $\epsilon_4 = \epsilon_4(\alpha, p) > 0$ such that*

(a) $$P\left(\sup_{s\geq 0, t\geq 2\epsilon} \sup_{u\leq 2\epsilon} |W_s(t-2\epsilon) - W_s(t-u)| \geq \alpha/2 \right) \leq p/8, \quad \forall \epsilon \leq \epsilon_4,$$

(b) $$P\left(\sup_{s\geq 2\epsilon} \sup_{|t-t'|\leq 2\epsilon} \sup_{\widetilde{y}\in\text{supp}(\widetilde{Y}^\epsilon_s)} \left|\widetilde{y}(t) - \widetilde{y}(t')\right| > \alpha/2 \right) \leq p/8, \quad \forall \epsilon \leq \epsilon_4 .$$

4 Construction of a pair of reflecting super-Brownian motions

Recall from Section 2 that $(\mathcal{T}^\epsilon, (l^\epsilon_u, u \in \mathcal{T}^\epsilon))$ is the marked genealogical forest corresponding to the ϵ-branching particle system Y^ϵ_t. Each element $u \in \mathcal{T}^\epsilon$ corresponds to a particle with the lifetime $l^\epsilon_u = \zeta^\epsilon_u - \xi^\epsilon_u$, where ζ^ϵ_u and ξ^ϵ_u are the death and birth times of the particle u respectively. The spatial motion of u is assumed to be a continuous function $f_u : [\xi^\epsilon_u, \zeta^\epsilon_u] \to \mathbb{R}$ and, moreover, $f_{u'}(\xi^\epsilon_{u'}) = f_u(\zeta^\epsilon_u)$ if u' is an offspring of u. Of course, in this paper, $\{f_u, u \in \mathcal{T}^\epsilon\}$ are Brownian paths. The historical path of u is the continuous function $w_u : [0, \zeta^\epsilon_u) \to \mathbb{R}$ such that for every $t \in [0, \zeta^\epsilon_u)$, $w_u(t)$ is the position at time t of the ancestor of u alive at that time. Let $l^\epsilon(\mathcal{T}^\epsilon) = \zeta^\epsilon(\mathcal{T}^\epsilon)$ be the lifetime of the ϵ-particle system Y^ϵ_t. For $\epsilon, t > 0$, Y^ϵ_t records the paths of (some) individuals up to time $t - 2\epsilon$. Hence,

$$\zeta^\epsilon(\mathcal{T}^\epsilon) = \left(\sup_{s\leq\tau} \beta_s - 2\epsilon \right)_+ .$$

We will need a truncation operator T_u acting on the forest. We define $T_u\mathcal{T}^\delta$ to be the subtree of $\mathcal{T}^\delta$ starting from the particle u.

We will use the "erasure of branches" idea of Neveu [5] to construct appropriately related ϵ- and δ-branching particle systems for any $\delta < \epsilon$. These in turn will be used to construct a δ-particle system with reflection from an ϵ-particle system with reflection.

Fix arbitrary $\epsilon > \delta > 0$. First we will just consider ϵ- and δ-marked branching forests corresponding to the particle systems without reflection—we will ignore the spatial motion of the particles. The following is the essence of Neveu's construction, but it can be also deduced from our historical process description. To pass from the δ-branching forest to the ϵ-branching forest one should erase each edge with no offspring (leaf) of the δ-forest from its endpoint to a point on the branch located at the distance $2(\epsilon-\delta)$ from the endpoint towards the root of the corresponding tree. If the length of that branch is more than $2(\epsilon-\delta)$ we cut it off by exactly $2(\epsilon-\delta)$. If the length of that edge is less than $2(\epsilon-\delta)$ we erase it completely and we proceed to the parent edge only when the neighboring edge (recall that the branching is binary) is also completely erased. The edges that have not been erased are then relabelled (edges with null lifetimes are excluded—this may change some marks) and this defines a marked ϵ-forest. More precisely, if $u \in \mathcal{T}^\delta$ satisfies

$$l^\delta(T_u\mathcal{T}^\delta) > 2(\epsilon-\delta) \tag{4.2}$$

then u belongs to the ϵ-forest after erasure and relabelling but it may have a different lifetime. Otherwise this particle and all its descendants are completely erased. For any $u \in \mathcal{T}^\delta$ define

$$l_u^{*,\delta} \equiv \min\left(l_u^\delta, (l^\delta(T_u\mathcal{T}^\delta) - 2(\epsilon-\delta))_+\right)$$

and

$$\mathcal{U}^1 \equiv \left\{u \in \mathcal{T}^\delta : l_u^{*,\delta} > 0,\ l_u^{*,\delta} = l^\delta(T_u\mathcal{T}^\delta) - 2(\epsilon-\delta)\right\}.$$

After relabelling, for any particle $u \in \mathcal{U}^1$ there is a particle $v_u \in \mathcal{T}^\epsilon$ with death time $\zeta^\epsilon_{v_u} = \xi^\delta_u + l_u^{*,\delta}$. Now recall (see Neveu-Pitman [6], [7]) that each death time of a particle $v_u \in \mathcal{T}^\epsilon$ corresponds to a local 2ϵ-maximum of the Brownian motion β. The branch $u \in \mathcal{U}^1$ is associated with a unique excursion e_u^δ of β on an interval (a_u, b_u), with the maximum above level $\xi_u^\delta + l_u^{*,\delta}$, i.e., $\beta(a_u) = \beta(b_u) = \xi_u^\delta + l_u^{*,\delta}$ and $\sup_{a_u \le s \le b_u} \beta(s) = \xi_u^\delta + l_u^{*,\delta} + 2\epsilon$.

For $u, v \in \mathcal{T}^\delta$, let $v < u$ mean that v is an ancestor of u, and

$$\mathcal{U}^2 \equiv \left\{u \in \mathcal{T}^\delta : l_u^{*,\delta} = 0,\ l_v^{*,\delta} = l_v^\delta,\ \forall v < u\right\} \bigcup \left\{u \in \mathcal{T}^\delta : l_u^{*,\delta} = 0,\ \xi_u^\delta = 0\right\}.$$

The first subset in the definition of $\mathcal{U}^2$ consists of the particles which are completely erased up to the parent level, but the subtree corresponding to their cousins is not completely erased, and hence their parents are not affected by the erasure. The second subset in the definition of $\mathcal{U}^2$ consists of the particles which are completely erased up to time zero (so, they do not have parents).

Again for any particle $u \in \mathcal{U}^2$ there is a subtree $T_u\mathcal{T}^\delta$ with life duration $l^\delta(T_u\mathcal{T}^\delta) < 2(\epsilon-\delta)$. Then there exists a unique local maximum of the Brownian motion β corresponding to this death time. Let e_u^δ be the unique excursion of β corresponding to $u \in \mathcal{U}^2$, on an interval (a_u, b_u) with the maximum above level ξ_u^δ, such that $\beta(a_u) = \beta(b_u) = \xi_u^\delta$ and $\sup_{a_u \le s \le b_u} \beta(s) = \xi_u^\delta + l^\delta(T_u\mathcal{T}^\delta) + 2\delta$.

Let $M = \left|\mathcal{U}^1 \cup \mathcal{U}^2\right|$ be the total number of elements in $\mathcal{U}^1 \cup \mathcal{U}^2$ and let

$$s_i = \beta(a_{u_i}), \quad i = 1, \ldots M,$$

be the "erasure levels" corresponding to the elements u_i of $\mathcal{U}^1 \cup \mathcal{U}^2$.

Recall from Section 3.1 of [1] that for any $\mathcal{T}^\epsilon$ there exists a forest $\widetilde{\mathcal{T}}^\epsilon$ representing the reflecting particle system. The historical paths $\widetilde{w}_{\widetilde{v}}$ are defined for $\widetilde{v} \in \widetilde{\mathcal{T}}^\epsilon$ in the same way as w_u are defined for $u \in \mathcal{T}^\epsilon$. Similarly, $\widetilde{\xi}^\epsilon_{\widetilde{v}}$ and $\widetilde{\zeta}^\epsilon_{\widetilde{v}}$ denote the times of birth and death of $\widetilde{v}$.

Lemma 4.1 *For every $i = 1, \ldots, M$, there exists a unique $\widetilde{v}_i \in \widetilde{\mathcal{T}}^\epsilon$ such that*

$$\widetilde{w}_{\widetilde{v}_i}(s_i) = W_{a_{u_i}}(s_i), \quad \text{for } \widetilde{\xi}^\epsilon_{\widetilde{v}_i} < s_i \le \widetilde{\zeta}^\epsilon_{\widetilde{v}_i}.$$

The proof of the lemma is elementary but tedious so it is left to the reader.

Next we will redefine the Brownian snake W on excursion intervals (a_{u_i}, b_{u_i}). For any $s \in (a_{u_i}, b_{u_i})$ let

$$\widehat{W}_s^{i,\delta}(t) = \begin{cases} W_s(t), & s_i \le t \le \beta_s, \\ \widetilde{w}^{\epsilon}_{\widetilde{v}_i}(t), & t \le s_i, \end{cases}$$

and let $\widehat{W}_s^{i,\delta,t}$ be the path of $\widehat{W}_s^{i,\delta}$ stopped at time t. Let $\widetilde{H}_t^{\epsilon}$ be the process analogous to the process H_t^{ϵ} defined in Section 2 but corresponding to the branching forest $\widetilde{\mathcal{T}}^{\epsilon}$ and the corresponding reflecting particle system. We will define a "historical" process representing the δ-particle system built on the top of the ϵ-reflecting particle system by adding some extra branches without reflection. Let

$$\widehat{H}_t^{\delta} = \widetilde{H}^{\epsilon}_{t+2(\epsilon-\delta)} + \sum_{i=1}^{M} \int_{a_{u_i}}^{b_{u_i}} \delta_{\widehat{W}_s^{i,\delta,t-2\delta}} L^t(ds). \tag{4.3}$$

Next we give mass δ^{-1} to each atom of $\widehat{H}_t^{\delta}$; this defines a historical measure valued process $\widehat{Y}_t^{\delta}$. The "historical" process $\widehat{Y}_t^{\delta}$ records the evolution of the δ-particle system up to time $t-2\delta$, with a special recipe for reflection. A particle alive at time $t-2\delta$ belongs to one of the following classes:

(i) particles "inherited" from the process $\widetilde{Y}^{\epsilon}$; these are the atoms of the historical measure $\widetilde{Y}^{\epsilon}_{t+2(\epsilon-\delta)}$ which describe the position and evolution of ϵ-reflecting particles which survived up to time $t-2\delta$;

(ii) particles that belong to the (non-reflecting) δ-system which are erased in the process of defining the ϵ-system from the δ-system.

Note that the particles in (ii) do not reflect with each other and do not reflect with particles in (i).
Define

$$\widehat{Y}_t^{i,\delta}(d\omega) = \mathbf{1}_{\{\omega(0)\in \mathrm{supp}(\mu_i)\}} \widehat{Y}_t^{\delta}(d\omega), \quad i = 1, 2.$$

Assumption. In what follows we fix $0 < p, \overline{\alpha} < 1$ arbitrary small and $a = \langle \mu, 1\rangle$. Then we choose $\nu^* \le \overline{\alpha}$ as in Lemma 3.1. For those $p, \overline{\alpha}, \nu^*$ we choose ϵ_1 as in Lemma 3.2. Then we choose $\alpha = \alpha(\nu^*, p), \epsilon_2 = \epsilon_2(\nu^*, p)$ and $\epsilon_3 = \epsilon_3(\overline{\alpha}, p)$ as in Lemma 3.4. Finally we choose $\epsilon_4 = \epsilon_4(\alpha, p)$ as in Corollary 3.6. Now fix some $\epsilon \le \min(\epsilon_i, i = 1, 2, 3, 4)$.

It is easy to see from our construction that if $\langle \widetilde{Y}_t^{i,\epsilon}, 1\rangle > 0$ then $\langle \widehat{Y}_t^{i,\delta}, 1\rangle > 0$, $i = 1, 2$. Define

$$\sigma^{i,\epsilon} \equiv \inf\{t \ge 2\epsilon : \ \langle \widetilde{Y}_t^{i,\epsilon}, 1\rangle = 0\}.$$

Let $\widehat{X}_t^{i,\delta}$ and $\widetilde{X}_t^{i,\epsilon}$ be the measure valued processes corresponding to $\widehat{Y}_t^{i,\delta}$ and $\widetilde{Y}_t^{i,\epsilon}$ in the same way as X_t^{ϵ} corresponds to Y_t^{ϵ}. For $t < \sigma^{1,\epsilon}$ let $\widehat{r}^{\delta}(t)$ (resp. $\widetilde{r}^{\epsilon}(t)$) be the right boundary of $\mathrm{supp}(\widehat{X}_t^{1,\delta})$ (resp. $\mathrm{supp}(\widetilde{X}_t^{1,\epsilon})$) and for $t < \sigma^{2,\epsilon}$ let $\widehat{l}^{\delta}(t)$ (resp. $\widetilde{l}^{\epsilon}(t)$) be the left boundary of $\mathrm{supp}(\widehat{X}_t^{2,\delta})$ (resp. $\mathrm{supp}(\widetilde{X}_t^{2,\epsilon})$).

Lemma 4.2 *For every $\delta \le \epsilon$,*

$$\begin{aligned} P\left(\sup_{2\epsilon \le t < \sigma^{1,\epsilon}} \left|\widehat{r}^\delta(t) - \widetilde{r}^\epsilon(t)\right| \le \alpha/2\right) &\ge 1 - p/4, \\ P\left(\sup_{2\epsilon \le t < \sigma^{2,\epsilon}} \left|\widehat{l}^\delta(t) - \widetilde{l}^\epsilon(t)\right| \le \alpha/2\right) &\ge 1 - p/4, \\ P\left(\sup_{2\epsilon \le t < \sigma^{1,\epsilon} \wedge \sigma^{2,\epsilon}} (\widehat{r}^\delta(t) - \widehat{l}^\delta(t)) \le \alpha\right) &\ge 1 - p/2. \end{aligned}$$

Proof By (4.3), the quantity $\sup_{2\epsilon \le t < \sigma^{1,\epsilon}} \left|\widehat{r}^\delta(t) - \widetilde{r}^\epsilon(t)\right|$ is bounded by the maximum of oscillations of the paths in the support of $\widetilde{Y}^\epsilon_t, t \ge 2\epsilon$ (described in Corollary 3.6 (b)), and the maximum of oscillations of extra paths added to the reflecting ϵ-branching system. The oscillations of the extra paths are bounded by oscillations of the Brownian snake process described in Corollary 3.6 (a). Hence, the first inequality follows from Corollary 3.6. The second inequality is analogous, and the third one follows from the first two and the fact that, by construction, $\widetilde{l}^\epsilon(t) \ge \widetilde{r}^\epsilon(t)$.

■

For each $i = 1, 2$, we relabel the paths of $\widehat{Y}^{i,\delta}_t$ to create a new "historical" process $\overline{Y}^{i,\delta}_t$ whose paths are reflecting from each other. Note that the paths are reflecting within each $\overline{Y}^{i,\delta}_t$, but the paths of $\overline{Y}^{1,\delta}_t$ are not reflecting from those in $\overline{Y}^{2,\delta}_t$. The corresponding measure-valued processes will be denoted $\overline{X}^{i,\delta}_t$. Let $\overline{r}^\delta(t)$ be the right boundary of $\text{supp}(\overline{X}^{1,\delta}_t)$ and $\overline{l}^\delta(t)$ be the left boundary of $\text{supp}(\overline{X}^{2,\delta}_t)$. Obviously $\overline{r}^\delta(t) = \widehat{r}^\delta(t)$ and $\overline{l}^\delta(t) = \widehat{l}^\delta(t)$.

Lemma 4.3 *For every $\delta \le \epsilon$,*

$$P\left(\sup_{2\epsilon \le t < \sigma^{1,\epsilon} \wedge \sigma^{2,\epsilon}} (\overline{r}^\delta(t) - \overline{l}^\delta(t)) \le \alpha\right) \ge 1 - p/2.$$

Proof The claim follows immediately from Lemma 4.2.

■

Let $(\overline{\mathcal{T}}^{i,\delta}, (\overline{l}_u, u \in \overline{\mathcal{T}}^{i,\delta}))$ be the marked tree corresponding to the genealogical structure of $\overline{Y}^{i,\delta}$, $i = 1, 2$. Note that by independence of the motion and the branching, $\overline{X}^{i,\delta}$ is the critical Galton-Watson branching process with the rate of branching δ^{-1}. Let $\overline{\beta}^{i,\delta}$, $i = 1, 2$, be the corresponding random walks. Note that the pair $(\overline{\beta}^{1,\delta}, \overline{\beta}^{2,\delta})$ has the same distribution as $(\beta^{1,\delta}, \beta^{2,\delta})$. Since

$$(\beta^{1,\delta}, \beta^{2,\delta}) \to (\beta^1, \beta^2)$$

in probability (see (2.3) in [1]), we obtain

$$(\overline{\beta}^{1,\delta}, \overline{\beta}^{2,\delta}) \to (\beta^1, \beta^2),$$

in distribution. By Lemma 3.2,

$$\sup_{i=1,2} P\left(\overline{L}^{i,\delta}(\overline{\beta}^{i,\delta}, \nu^*) \ge 2\overline{\alpha}\right) \le p/4, \quad \forall \delta \le \epsilon, \tag{4.4}$$

where $\overline{L}^{i,\delta}$ is defined relative to $\overline{\beta}^{i,\delta}$ in the same way as $L^{i,\delta}$ was defined relative to β^δ in Section 3.

Let

$$\mathbf{T}^i \equiv \left\{t \geq 2\epsilon : \ \langle \widetilde{X}^{i,\epsilon}_t, 1\rangle \geq 4\overline{\alpha}\right\}, \quad i = 1,2,$$

$$\overline{\eta}^{\overline{\alpha}}_1(t) = \begin{cases} \sup\{x : \ \int_x^\infty \widetilde{X}^{1,\epsilon}_t(dy) \geq 4\overline{\alpha}\}, & \forall t \in \mathbf{T}^1, \\ -\infty, & \forall t \notin \mathbf{T}^1, \end{cases}$$

$$\overline{\eta}^{\overline{\alpha}}_2(t) = \begin{cases} \inf\{x : \ \int_{-\infty}^x \widetilde{X}^{2,\epsilon}_t(dy) \geq 4\overline{\alpha}\}, & \forall t \in \mathbf{T}^2, \\ \infty, & \forall t \notin \mathbf{T}^2, \end{cases}$$

$$\begin{aligned} \Gamma^{1,\delta,\epsilon} &= \left\{\omega : \ \overline{X}^{2,\delta}_t \left([\overline{\eta}^{\overline{\alpha}}_1(t), \widetilde{r}^\epsilon(t) - \alpha/2]\right) = 0, \ \forall t \in \mathbf{T}^1\right\} \\ &\cap \left\{\omega : \ \sup_{t\geq 2\epsilon} \sup_{x\in\mathbb{R}, r\geq 0} \left|X^\epsilon_t(B(x,r)) - X^\delta_t(B(x,r))\right| \leq \overline{\alpha}\right\} \\ &\cap \left\{\omega : \ X^\epsilon_t\left([\widetilde{r}^\epsilon(t) - \alpha/2, \widetilde{r}^\epsilon(t)]\right) \leq \nu^*/2, \ \forall t \in \mathbf{T}^1\right\}, \end{aligned} \tag{4.5}$$

$$\begin{aligned} \Gamma^{2,\delta,\epsilon} &= \left\{\omega : \ \overline{X}^{1,\delta}_t \left([\widetilde{l}^\epsilon(t) + \alpha/2, \overline{\eta}^{\overline{\alpha}}_2(t)]\right) = 0, \ \forall t \in \mathbf{T}^2\right\} \\ &\cap \left\{\omega : \ \sup_{t\geq 2\epsilon} \sup_{x\in\mathbb{R}, r\geq 0} \left|X^\epsilon_t(B(x,r)) - X^\delta_t(B(x,r))\right| \leq \overline{\alpha}\right\} \\ &\cap \left\{\omega : \ X^\epsilon_t\left([\widetilde{l}^\epsilon(t), \widetilde{l}^\epsilon(t) + \alpha/2]\right) \leq \nu^*/2, \ \forall t \in \mathbf{T}^2\right\}. \end{aligned} \tag{4.6}$$

By Lemmas 3.4 and 4.2, we obtain for $i = 1,2$,

$$\begin{aligned} P\left(\Gamma^{i,\delta,\epsilon}\right) &\geq 1 - p/4 - p/2 - p/4 \\ &= 1 - p. \end{aligned} \tag{4.7}$$

Let $\overline{X}^\delta_t = \overline{X}^{1,\delta}_t + \overline{X}^{2,\delta}_t$. It follows from (4.5) and the definitions of $\widetilde{r}^\epsilon(\cdot)$ and $\overline{\eta}^{\overline{\alpha}}_1(\cdot)$ that for any $\omega \in \Gamma^{1,\delta,\epsilon}$ and $t \in \mathbf{T}^1$,

$$\begin{aligned} \overline{X}^{1,\delta}_t \left([\overline{\eta}^{\overline{\alpha}}_1(t), \infty)\right) &\geq \overline{X}^{1,\delta}_t \left([\overline{\eta}^{\overline{\alpha}}_1(t), \widetilde{r}^\epsilon(t) - \alpha/2]\right) \\ &= \overline{X}^{\delta}_t \left([\overline{\eta}^{\overline{\alpha}}_1(t), \widetilde{r}^\epsilon(t) - \alpha/2]\right) \\ &\geq X^{\epsilon}_t \left([\overline{\eta}^{\overline{\alpha}}_1(t), \widetilde{r}^\epsilon(t) - \alpha/2]\right) - \overline{\alpha} \\ &= X^{\epsilon}_t \left([\overline{\eta}^{\overline{\alpha}}_1(t), \widetilde{r}^\epsilon(t)]\right) - X^\epsilon_t\left((\widetilde{r}^\epsilon(t) - \alpha/2, \widetilde{r}^\epsilon(t)]\right) - \overline{\alpha} \\ &\geq \widetilde{X}^{1,\epsilon}_t \left([\overline{\eta}^{\overline{\alpha}}_1(t), \widetilde{r}^\epsilon(t)]\right) - \nu^*/2 - \overline{\alpha} \\ &\geq 2\overline{\alpha}. \end{aligned} \tag{4.8}$$

In a similar way we get that for any $\omega \in \Gamma^{2,\delta,\epsilon}$ and $t \in \mathbf{T}^2$,

$$\overline{X}^{2,\delta}_t \left((-\infty, \overline{\eta}^{\overline{\alpha}}_2(t)]\right) \geq 2\overline{\alpha}.$$

We proceed to do one more (final!) relabelling of the particles. We relabel the particles of $\overline{Y}^{1,\delta} \cup \overline{Y}^{2,\delta}$ in such a way that all paths are reflecting from each other. We group the relabelled paths into two families $\widetilde{Y}^{1,\delta}$ and $\widetilde{Y}^{2,\delta}$ so that the roots of trees in $\widetilde{Y}^{i,\delta}$ are the same as in $\overline{Y}^{i,\delta}$. Note that we are using the same notation $\widetilde{Y}^{i,\delta}$ as in Section 2. This is no coincidence—the pair of processes just defined is the same as $(\widetilde{Y}^{1,\delta},\widetilde{Y}^{2,\delta})$ of Section 2, since the intermediate labelling scheme of $\overline{Y}^{i,\delta}$ has no effect on the final result.

Define $\widetilde{S}^{i,\delta}(t)$ (resp. $\overline{S}^{i,\delta}(t)$) to be the collection of atoms of $\widetilde{Y}^{i,\delta}$ (resp. $\overline{Y}^{i,\delta}$), $i=1,2$. Also let

$$\widetilde{S}^{1,\delta,\overline{\alpha}}(t)=\left\{\widetilde{y}(\cdot)\in\widetilde{S}^{1,\delta}(t):\ \widetilde{y}(t)\le\eta_1^{\overline{\alpha}}(t)\right\},$$

$$\widetilde{S}^{2,\delta,\overline{\alpha}}(t)=\left\{\widetilde{y}(\cdot)\in\widetilde{S}^{2,\delta}(t):\ \widetilde{y}(t)\ge\eta_2^{\overline{\alpha}}(t)\right\},$$

$$\overline{S}^{1,\delta,\overline{\alpha}}(t)=\left\{\overline{y}(\cdot)\in\overline{S}^{1,\delta}(t):\ \overline{y}(t)\le\overline{\eta}_1^{\overline{\alpha}}(t)\right\},$$

$$\overline{S}^{2,\delta,\overline{\alpha}}(t)=\left\{\overline{y}(\cdot)\in\overline{S}^{2,\delta}(t):\ \overline{y}(t)\ge\overline{\eta}_2^{\overline{\alpha}}(t)\right\},$$

$$\begin{aligned}A^{1,\delta}&\equiv\left\{\forall t\in\mathbf{T}^1,\forall y\in\overline{S}^{1,\delta,\overline{\alpha}}(t):\ y(s)\le\widetilde{r}^{\epsilon}(s)-\alpha/2,\forall s\le t\right\},\\ A^{2,\delta}&\equiv\left\{\forall t\in\mathbf{T}^2,\forall y\in\overline{S}^{2,\delta,\overline{\alpha}}(t):\ y(s)\ge\widetilde{l}^{\epsilon}(s)+\alpha/2,\forall s\le t\right\}.\end{aligned}$$

Lemma 4.4 *We have for $i=1,2$,*

$$P\left(A^{i,\delta}\cap\Gamma^{i,\delta,\epsilon}\right)\ge1-\frac{7}{4}p.$$

Proof

$$\begin{aligned}P\left(\Gamma^{1,\delta,\epsilon}\cap\left(A^{1,\delta}\right)^c\right)&=P\left(\Gamma^{1,\delta,\epsilon}\cap\left\{\exists t\in\mathbf{T}^1,y\in\overline{S}^{1,\delta,\overline{\alpha}}(t):\ y(s)>\widetilde{r}^{\epsilon}(s)-\alpha/2,\ \text{for some } s\le t\right\}\right)\\
&\le P\left(\Gamma^{1,\delta,\epsilon}\cap\left\{\exists t\in\mathbf{T}^1,y\in\overline{S}^{1,\delta,\overline{\alpha}}(t):\ y(s)>\overline{r}^{\delta}(s)-\alpha,\ \text{for some } s\le t\right\}\right)\\
&\quad+p/4\\
&\le P\left(\Gamma^{1,\delta,\epsilon}\cap\left\{\exists t\in\mathbf{T}^1,y\in\overline{S}^{1,\delta,\overline{\alpha}}(t):\ y(s)>\overline{r}^{\delta}(s)-\alpha,\ \text{for some } s\le t\right\}\right.\\
&\quad\left.\cap\left\{\overline{X}^{1,\delta}_u\left([\overline{r}^{\delta}(u)-\alpha,\overline{r}^{\delta}(u)]\right)\le\nu^*/2,\ \forall u\ge2\delta\right\}\right)+p/2\\
&\le P\left(\Gamma^{1,\delta,\epsilon}\cap\left\{\overline{L}^{1,\delta}\left(\overline{\beta}^{1,\delta},\nu^*/2\right)\ge2\overline{\alpha}\right\}\right.\\
&\quad\left.\cap\left\{\overline{X}^{1,\delta}_u\left([\overline{r}^{\delta}(u)-\alpha,\overline{r}^{\delta}(u)]\right)\le\nu^*/2,\ \forall u\ge2\delta\right\}\right)+p/2\\
&\le p/4+p/2=3p/4,\end{aligned}$$

The first inequality follows by Lemma 4.2, the second by Lemma 3.4 and the last inequality follows by (4.4). The third inequality, which follows by (4.8), is crucial, in a sense — various objects have been defined to make the event on the left hand side a subset of the event on the right hand side.

The inequality $P\left(\Gamma^{2,\delta,\epsilon}\cap\left(A^{2,\delta}\right)^c\right)\le3p/4$ follows along the same lines. This and (4.7) yield the desired result.

■

Lemma 4.5 *For $i=1,2$, and any $\omega\in\Gamma^{i,\delta,\epsilon}\cap A^{i,\delta}$,*

$$\widetilde{S}^{i,\delta,\overline{\alpha}}(t)(\omega)=\overline{S}^{i,\delta,\overline{\alpha}}(t)(\omega),\ \forall t\in\mathbf{T}^i,$$

and hence, by Lemma 4.4,

$$P\left(\widetilde{S}^{i,\delta,\overline{\alpha}}(t)=\overline{S}^{i,\delta,\overline{\alpha}}(t),\ \forall t\in\mathbf{T}^i\right)\ge1-\frac{7}{4}p.$$

Proof We will discuss only the case $i = 1$, since the proof for $i = 2$ is analogous. Assume that $\omega \in \Gamma^{1,\delta,\epsilon} \cap A^{1,\delta}$, fix an arbitrary $t \in \mathbf{T}^1$ and $y \in \overline{S}^{1,\delta,\overline{\alpha}}(t)$. Recall that all the paths inside $\overline{S}^{1,\delta,\overline{\alpha}}(t)$ are reflecting. Hence y is not crossed by any path from any family $\overline{S}^{1,\delta,\overline{\alpha}}(s)$, for any $s \le t$. If after the final relabelling $y \notin \widetilde{S}^{1,\delta,\overline{\alpha}}(t)$, it means that y is intersected by a path in the family $\bigcup_{s\le t} \overline{S}^{2,\delta,\overline{\alpha}}(s)$. This combined with the first line in the definition of $\Gamma^{1,\delta,\epsilon}$ shows that

$$\exists s: \; y(s) \ge \widetilde{r}^{\epsilon}(s) - \alpha/2.$$

This however contradicts the fact that $\omega \in A^{1,\delta}$. Therefore

$$\overline{S}^{1,\delta,\overline{\alpha}}(t) \subset \widetilde{S}^{1,\delta,\overline{\alpha}}(t), \qquad \forall t \in \mathbf{T}^1, \omega \in \Gamma^{1,\delta,\epsilon} \cap A^{1,\delta}.$$

Conversely, suppose that $y \in \widetilde{S}^{1,\delta,\overline{\alpha}}(t)$. Recall that for any $t \ge 2\delta$ and $i = 1, 2$, $\widetilde{Y}_t^{i,\delta}$ and $\overline{Y}_t^{i,\delta}$ record the evolution of the corresponding δ-particle sytems up to time $t_\delta \equiv t - 2\delta$. Hence the paths in $\widetilde{S}^{1,\delta,\overline{\alpha}}(t)$ and $\overline{S}^{1,\delta,\overline{\alpha}}(t)$ are the paths stopped at time t_δ. There is only a finite number of branching times in the δ-particle system. We will assume without loss of generality that t_δ is not a branching time — it is easy to see that the result follows for all times if we prove the assertion for only non-branching times t_δ.

The fact that $\omega \in \Gamma^{1,\delta,\epsilon} \cap A^{1,\delta}$ implies that the functions in $\overline{S}^{1,\delta,\overline{\alpha}}(t)$ evaluated at t_δ mark the positions of all particles in the δ-system located at or to the left of $\overline{\eta}_1^{\overline{\alpha}}(t)$. But these are the same locations as those of particles corresponding to $\widetilde{S}^{1,\delta,\overline{\alpha}}(t)$, at time t_δ. Since t_δ is not a branching time, all the paths in $\overline{S}^{1,\delta,\overline{\alpha}}(t)$ and $\widetilde{S}^{1,\delta,\overline{\alpha}}(t)$ coincide on $[t', t_\delta]$, for some $t' < t_\delta$. Hence any $y \in \widetilde{S}^{1,\delta,\overline{\alpha}}(t)$ coincides with some $\overline{y} \in \overline{S}^{1,\delta,\overline{\alpha}}(t)$ on $[t', t_\delta]$. By the first part of the proof, $\overline{y} \in \widetilde{S}^{1,\delta,\overline{\alpha}}(t)$. Thus we have

$$\exists y, \overline{y} \in \widetilde{S}^{1,\delta,\overline{\alpha}}(t)$$

such that

$$y(s) = \overline{y}(s), \;\; \forall s \in [t', t_\delta].$$

Since $\widetilde{S}^{1,\delta,\overline{\alpha}}(t)$ has a tree structure and all the paths in $\widetilde{S}^{1,\delta,\overline{\alpha}}(t)$ are stopped at time t_δ, we obtain

$$y(s) = \overline{y}(s), \;\; \forall s \in [0, t].$$

Now recall that $\overline{y} \in \overline{S}^{1,\delta,\overline{\alpha}}(t)$ to see that $y \in \overline{S}^{1,\delta,\overline{\alpha}}(t)$ and hence

$$\widetilde{S}^{1,\delta,\overline{\alpha}}(t) \subset \overline{S}^{1,\delta,\overline{\alpha}}(t), \qquad \forall t \in \mathbf{T}^1, \omega \in \Gamma^{1,\delta,\epsilon} \cap A^{1,\delta}.$$

This completes the proof of the lemma.

■

Corollary 4.6 *For any* $\omega \in \bigcap_{i=1,2}(\Gamma^{i,\delta,\epsilon} \cap A^{i,\delta})$,

$$\widetilde{X}_t^{1,\delta}\left([\overline{\eta}_2^{\overline{\alpha}}(t), \infty)\right) = 0, \qquad \widetilde{X}_t^{2,\delta}\left((-\infty, \overline{\eta}_1^{\overline{\alpha}}(t)]\right) = 0, \;\; \forall t \ge 2\epsilon.$$

Proof Fix arbitrary $t \ge 2\epsilon$. For $t \in \mathbf{T}^1 \cap \mathbf{T}^2$ the result follows from the previous lemma. For $t \notin \mathbf{T}^1 \cap \mathbf{T}^2$ the result follows by definition of $\overline{\eta}_i^{\overline{\alpha}}(t)$, $i = 1, 2$, and again by the previous lemma.

■

Lemma 4.7

$$P\left(\sup_{t\ge 2\epsilon}\left|\langle \widetilde{X}_t^{i,\epsilon}, 1\rangle - \langle \widetilde{X}_t^{i,\delta}, 1\rangle\right| \ge 14\overline{\alpha}\right) \le \frac{14}{4}p, \qquad i = 1, 2.$$

Proof Consider the case $i=1$, and assume that $\omega \in \bigcap_{i=1,2}\left(\Gamma^{i,\delta,\epsilon}\cap A^{i,\delta}\right)$. Then for all $t\geq 2\epsilon$,

$$\begin{aligned}
\left|\langle\widetilde{X}_t^{1,\epsilon},1\rangle-\langle\widetilde{X}_t^{1,\delta},1\rangle\right| &\leq \left|\widetilde{X}_t^{1,\epsilon}\left((-\infty,\overline{\eta}_1^{\overline{\alpha}}(t)]\right)-\widetilde{X}_t^{1,\delta}\left((-\infty,\overline{\eta}_1^{\overline{\alpha}}(t)]\right)\right| \\
&\quad+\left|\widetilde{X}_t^{1,\epsilon}\left((\overline{\eta}_1^{\overline{\alpha}}(t),\infty)\right)\right|+\left|\widetilde{X}_t^{1,\delta}\left((\overline{\eta}_1^{\overline{\alpha}}(t),\infty)\right)\right| \\
&\leq \left|\widetilde{X}_t^{\epsilon}\left((-\infty,\overline{\eta}_1^{\overline{\alpha}}(t)]\right)-\widetilde{X}_t^{\delta}\left((-\infty,\overline{\eta}_1^{\overline{\alpha}}(t)]\right)\right|+4\overline{\alpha}+\left|\widetilde{X}_t^{1,\delta}\left((\overline{\eta}_1^{\overline{\alpha}}(t),\overline{\eta}_2^{\overline{\alpha}}(t))\right)\right| \\
&\leq 5\overline{\alpha}+\left|\widetilde{X}_t^{\delta}\left((\overline{\eta}_1^{\overline{\alpha}}(t),\overline{\eta}_2^{\overline{\alpha}}(t))\right)\right| \\
&\leq 5\overline{\alpha}+\overline{\alpha}+\left|\widetilde{X}_t^{\epsilon}\left((\overline{\eta}_1^{\overline{\alpha}}(t),\overline{\eta}_2^{\overline{\alpha}}(t))\right)\right| \\
&\leq 6\overline{\alpha}+8\overline{\alpha}=14\overline{\alpha},
\end{aligned}$$

where the second inequality follows by the definition of $\overline{\eta}_1^{\overline{\alpha}}(t)$ and Corollary 4.6, the third one by the definition of $\Gamma^{1,\delta,\epsilon}$, the fourth one is again by the definition of $\Gamma^{1,\delta,\epsilon}$ and the last one by the definition of $\overline{\eta}_i^{\overline{\alpha}}(t), i=1,2$. A similar inequality holds for $i=2$. The result follows now from Lemma 4.4.

■

Proof of Theorem 2.2 By Lemma 4.7, $\{(\langle\widetilde{X}^{1,\epsilon},1\rangle,\langle\widetilde{X}^{2,\epsilon},1\rangle),\epsilon\leq 1\}$ is a Cauchy family, in the sense of convergence in probability. On the other hand, $X^\epsilon=\widetilde{X}^{1,\epsilon}+\widetilde{X}^{2,\epsilon}$ converges in D_{M_F} to X, in probability. This, and the fact that for each t, the support of $\widetilde{X}_t^{1,\epsilon}$ lies to the left of the support of $\widetilde{X}_t^{2,\epsilon}$ implies that $(\widetilde{X}^{1,\epsilon},\widetilde{X}^{2,\epsilon})$ converges in $D_{M_F\times M_F}$ in probability, as $\epsilon\downarrow 0$.

■

Next we are going to extend the above result to define reflecting super-Brownian motions starting at any time $s\geq 0$. Fix an arbitrary $s\geq 0$, and let $\widetilde{X}_s^{1,s}$ and $\widetilde{X}_s^{2,s}$ be any random measures in M_F such that

1. $\widetilde{X}_s^{1,s}+\widetilde{X}_s^{2,s}=X_s$.
2. If $\langle\widetilde{X}_s^{1,s},1\rangle>0,\langle\widetilde{X}_s^{2,s},1\rangle>0$ then the support of $\widetilde{X}_s^{1,s}$ lies to the left of the support of $\widetilde{X}_s^{2,s}$, that is for any $x_i\in\mathrm{supp}(\widetilde{X}_s^{i,s})$, $i=1,2$, we have $x_1\leq x_2$.
3. $(\widetilde{X}_s^{1,s},\widetilde{X}_s^{2,s})\in\mathcal{F}_s{}^X:=\sigma\{X_t\,,t\leq s\}$.

We will consider truncations of all superprocesses, historical processes with and without reflection, etc. to the time interval $[s,\infty)$ (the related ϵ-approximations of superprocesses and historical processes will be truncated to the time interval $[s+2\epsilon,\infty)$). Let $\{Y_t^{s,\epsilon},t\geq s+2\epsilon\}$ be the natural truncation of $\{Y_t^\epsilon,t\geq 2\epsilon\}$, and note that it is the same as the approximating branching system constructed on the basis of the historical process $\{Y_t,t\geq 0\}$ truncated to the interval $[s,\infty)$. In other words, one obtains the same process $\{Y_t^{s,\epsilon},t\geq s+2\epsilon\}$, no matter which operation is performed first on $\{Y_t,t\geq 0\}$ — truncation to the interval $[s,\infty)$ or passing to the semi-discrete approximation. Let $\{(\widetilde{X}_t^{1,s,\epsilon},\widetilde{X}_t^{2,s,\epsilon}),\ t\geq s+2\epsilon\}$ be a pair of approximating reflecting super-Brownian motions defined as in Section 2 but this time relative to $\{Y_t^{s,\epsilon},t\geq s+2\epsilon\}$. The processes start at time $t=s+2\epsilon$ at

$$\widetilde{X}_{s+2\epsilon}^{i,s,\epsilon}(dx)\ =\ \mathbf{1}_{\{x\in\mathrm{supp}(\widetilde{X}_s^{i,s})\}}X_{s+2\epsilon}^\epsilon(dx),\ \ i=1,2.$$

Proposition 4.8 *There exists a pair of processes* $\{(\widetilde{X}_t^{1,s}, \widetilde{X}_t^{2,s}),\ t \geq s\}$ *in* $C_{M_F \times M_F}[s,\infty)$ *("reflecting super-Brownian motions") such that*

$$(\widetilde{X}^{1,s,\epsilon}, \widetilde{X}^{2,s,\epsilon}) \to (\widetilde{X}^{1,s}, \widetilde{X}^{2,s}),$$

as $\epsilon \downarrow 0$*, in probability in* $D_{M_F \times M_F}[s,\infty)$*.*

Proof For any $\overline{\alpha} > 0$, for P-a.s. ω,

$$\lim_{\epsilon,\delta\downarrow 0, \epsilon>\delta} P\left(\sup_{t\geq s+2\epsilon} \left| \langle \widetilde{X}_t^{i,s,\epsilon}, 1\rangle - \langle \widetilde{X}_t^{i,s,\delta}, 1\rangle \right| \geq \overline{\alpha} \,\middle|\, \widetilde{X}_s^{1,s}, \widetilde{X}_s^{2,s} \right)(\omega) = 0, \qquad i = 1,2,$$

by Lemma 4.7. Hence, by dominated convergence we obtain that

$$\lim_{\epsilon,\delta\downarrow 0, \epsilon>\delta} P\left(\sup_{t\geq s+2\epsilon} \left| \langle \widetilde{X}_t^{i,s,\epsilon}, 1\rangle - \langle \widetilde{X}_t^{i,s,\delta}, 1\rangle \right| \geq \overline{\alpha} \right) = 0, \qquad i = 1,2,$$

and then we can proceed as in the proof of Theorem 2.2.

■

We will generalize the last result even further by passing from pairs to families of reflecting super-Brownian motions. For a real $s \geq 0$ and integers $i \in \mathbb{Z}$, $n \geq 1$, let $\mu^{i,s,n} \in M_F$ be the truncation of X_s to $[i2^{-n}, (i+1)2^{-n})$, i.e.,

$$\mu^{i,s,n}(A) = X_s(A \cap [i2^{-n}, (i+1)2^{-n})), \quad A \subset \mathbb{R}.$$

For a fixed $s \geq 0$ and $\epsilon > 0$, let $\{(\widetilde{X}_t^{i,s,\epsilon,n},\ t \geq s + 2\epsilon, i \in \mathbb{Z}\}$ be the family of approximating reflecting super-Brownian motions constructed from $\{Y_t^{s,\epsilon}, t \geq s+2\epsilon\}$ in a way similar to that in Section 2, with different processes starting at time $t = s+2\epsilon$ from

$$\widetilde{X}_{s+2\epsilon}^{i,s,\epsilon,n}(dx) \;=\; \mathbf{1}_{\{x\in \mathrm{supp}(\mu^{i,s,n})\}} X_{s+2\epsilon}^{\epsilon}(dx), \;\; i \in \mathbb{Z}.$$

Theorem 4.9 *Fix an* $s \geq 0$*. For each* $n \geq 1$ *there exists a family of reflecting super-Brownian motions* $\{\widetilde{X}_t^{i,s,n}, t \geq s,\ i \in \mathbb{Z}\}$ *such that*

- **(a)** $\widetilde{X}_s^{i,s,n} = \mu^{i,s,n}$, $\forall i \in \mathbb{Z}$.
- **(b)** $\lim_{\epsilon\downarrow 0} \left(\widetilde{X}^{i,s,\epsilon,n}\right)_{i\in\mathbb{Z}} = \left(\widetilde{X}^{i,s,n}\right)_{i\in\mathbb{Z}}$*, in* $(D_{M_F}[s,\infty))^\infty$*, in probability.*
- **(c)** *For any* $t \geq s, i \in \mathbb{Z}$*, such that* $\widetilde{X}_t^{i,s,n}, \widetilde{X}_t^{i+1,s,n} > 0$*, the support of* $\widetilde{X}_t^{i,s,n}$ *lies to the left of the support of* $\widetilde{X}_t^{i+1,s,n}$*.*

Proof The theorem is a straightforward extension of Proposition 4.8.

■

5 Proof of Theorem 2.1

Recall our conventions from the "Notation" section at the end of the Introduction. For any $x \in \mathbb{R}$ define $[x]_n = \frac{[x2^n]}{2^n}$. For any $k, n \geq 1$ consider the following family of functions:

$$D^{k,n} = \left\{ \psi \in B\left(\mathbb{R}^k\right) : \psi(x_1, \ldots, x_k) = \psi([x_1]_n, \ldots, [x_k]_n), \ \forall (x_1, \ldots, x_k) \in \mathbb{R}^k \right\}.$$

In words, $D^{k,n}$ consists of functions which are constant on intervals of the form $[\frac{i_1}{2^n}, \frac{i_1+1}{2^n}) \times \ldots \times [\frac{i_k}{2^n}, \frac{i_k+1}{2^n})$. Next we define some families of functions on $C_{\mathbb{R}}[0,\infty)$:

$$\begin{aligned} D_t &= \Big\{ \psi \in C(C_{\mathbb{R}}[0,\infty)) : \psi(y) = \overline{\psi}(y(t_1), \ldots, y(t_k)) \text{ for some } \overline{\psi} \in C_u(\mathbb{R}^k), \\ &\qquad 0 \leq t_1 < \ldots < t_k < t, \ k \geq 1 \Big\}, \\ D_t^{k,n} &= \Big\{ \psi \in C(C_{\mathbb{R}}[0,\infty)) : \psi(y) = \overline{\psi}(y(t_1), \ldots, y(t_k)) \text{ for some } \overline{\psi} \in D^{k,n} \Big\}. \end{aligned}$$

The following two auxiliary lemmas will help us complete the proof of Theorem 2.1.

Lemma 5.1 *Fix arbitrary $t > 0$. Let $\{Z^\epsilon, \ \epsilon \leq 1\}$ be a family of $M_F(C[0,t])$-valued random variables, whose laws are tight in the space of all probability measures on $M_F(C[0,t])$. Suppose for each $k, n \geq 1$ and $\psi \in D_t^{k,n}$ there exists a random variable Z^ψ such that $\langle Z^\epsilon, \psi \rangle \to Z^\psi$, as $\epsilon \downarrow 0$, in probability. Then there exists a random measure $Z \in M_F(C[0,t])$ such that*

$$Z^\epsilon \to Z, \text{ as } \epsilon \downarrow 0,$$

in $M_F(C[0,t])$, in probability.

Lemma 5.2 *Let $\{\widetilde{Y}^\epsilon, \ \epsilon \leq 1\}$ be a family of processes in $D_{M_F(C)}[0,\infty)$ whose laws are tight in the space of all probability measures on $D_{M_F(C)}[0,\infty)$, and any limit law is supported by $C_{M_F(C)}[0,\infty)$. If for each $t > 0$, $\widetilde{Y}^\epsilon_t$ converges in $M_F(C)$, in probability, then there exists a process $\widetilde{Y} \in C_{M_F(C)}[0,\infty)$ such that*

$$\widetilde{Y}^\epsilon \to \widetilde{Y}, \quad \text{as } \epsilon \downarrow 0,$$

in $D_{M_F(C)}[0,\infty)$, in probability.

Proof of Lemma 5.1 We adopt the argument which is used for the proof of Lemma A.5 of [2]. We will argue by contradiction. Let d be a metric on $M_F(C[0,t])$. Suppose that there exist $\eta > 0, M > 0, \{\epsilon_m, \delta_m, m \geq 1\}$ such that $\epsilon_m, \delta_m \downarrow 0$, as $m \to \infty$, and

$$P\left(d(Z^{\epsilon_m}, Z^{\delta_m}) \geq \eta\right) \geq \eta, \ \forall m \geq M.$$

By our assumptions we can choose a subsequence $(Z^{\epsilon'_m}, Z^{\delta'_m})$ which converges in $M_F(C[0,t]) \times M_F(C[0,t])$ in law. Let (Z', Z'') be its limit point defined possibly on another probability space. Now note that the set of functions $\{D_t^{k,n}, k, n \geq 1\}$ is dense in D_t in the uniform topology. Hence we easily see that,

$$\lim_{m\to\infty} P\left(\left| \langle Z^{\epsilon'_m}, \psi \rangle - \langle Z^{\delta'_m}, \psi \rangle \right| \geq \gamma \right) = 0, \ \ \forall \psi \in D_t, \ \gamma > 0.$$

Since the functions in D_t are continuous on $M_F(C[0,t])$ and D_t separates measures in $M_F(C[0,t])$ we immediately get that

$$Z' = Z'', \ \ P-\text{a.s.}.$$

But this implies that

$$\lim_{m\to\infty} P\left(d(Z^{\epsilon'_m}, Z^{\delta'_m}) \geq \eta\right) = 0,$$

yielding the contradiction and we are done.

■

Proof of Lemma 5.2 The proof of Lemma 5.2 goes along the same lines as the proof of Lemma A.5 of [2] (or Lemma 5.1) and hence is omitted.

■

Fix arbitrary $t > 0$, $n \geq 1$, and $0 \leq s_1 < \ldots < s_K < t$. Let $\{\widetilde{X}_s^{i,s_k,n}, s \geq s_k,\ i \in \mathbb{Z}, k = 1,\ldots,K\}$ be the family of reflecting super-Brownian motions constructed in Theorem 4.9. Let $\widetilde{l}^{k,i,n,t}$ ($\widetilde{r}^{k,i,n,t}$) be the left (right) endpoint of the support of $\widetilde{X}_t^{i,s_k,n}$ defined whenever $\langle \widetilde{X}_t^{i,s_k,n}, 1\rangle > 0$. Note that we always have $\widetilde{r}^{k,i,n,t} \leq \widetilde{l}^{k,i+1,n,t}$ but not necessarily $\widetilde{r}^{k,i,n,t} = \widetilde{l}^{k,i+1,n,t}$. Define

$$I_k \equiv \{i \in \mathbb{Z} : \langle \widetilde{X}_t^{i,s_k,n}, 1\rangle > 0\}$$

and let $|I_k|$ be the total number of elements in I_k. Let $M_{n,t}$ be the set of all real numbers $\widetilde{l}^{k,i,n,t}$ and $\widetilde{r}^{k,i,n,t}$ for $k = 1, 2, \ldots, K$, $i \in I_k$.

Lemma 5.3 *With probability* 1*, only a finite number of processes in the family* $\{\widetilde{X}^{i,s_k,n}(s), s \geq s_k,\ i \in \mathbb{Z}, k = 1,\ldots,K\}$ *survive up to time* $t > s_K$*, that is,*

$$\sum_{k=1}^{K} |I_k| < \infty, \qquad P-\text{a.s.}$$

Proof There exists only a finite number of excursions of reflecting Brownian motion starting at any level s_k, $k = 1,\ldots,K$ which reach level $t > s_K$, a.s. The lemma now follows immediately from the snake representation of the reflecting super-Brownian motion.

■

Proposition 5.4 *Let* $\widetilde{Y}^{\epsilon}$ *be as in Theorem 2.1. For any* $K, n \geq 1$, $t > 0$, $\psi \in D_t^{K,n}$,

$$P\left(\left|\langle \widetilde{Y}_t^{\epsilon_1}, \psi\rangle - \langle \widetilde{Y}_t^{\epsilon_2}, \psi\rangle\right| > \eta\right) \to 0, \quad \text{as } \epsilon_1, \epsilon_2 \downarrow 0,\ \forall \eta > 0.$$

Proof Fix an arbitrary $\psi \in D_t^{K,n}$, and $\eta, p > 0$ arbitrary small. Let $A(\gamma) = \{x \in \mathbb{R} : \operatorname{dist}(x, M_{n,t}) > \gamma\}$. Recall that X_t has a continuous density and hence, by Lemma 5.3, we can find $\gamma_1 > 0$ so small that for $\gamma \leq \gamma_1$, $P(X_t(A(\gamma)^c) > \eta/(4\|\psi\|_\infty)) < p$. By Theorem 4.9 and reflection properties of $\widetilde{X}^{i,s_k,\epsilon,n}$ we immediately get that for each $k = 1,\ldots,K$ and $i \in \mathbb{Z}$,

$$\lim_{\epsilon\downarrow 0} P\left(\widetilde{X}_t^{j,s_k,\epsilon,n}((\widetilde{l}^{k,i,n,t} + \gamma, \widetilde{r}^{k,i,n,t} - \gamma)) > 0 \,\Big|\, \langle \widetilde{X}_t^{i,s_k,n}, 1\rangle > 0\right) = 0, \quad \forall j \neq i.$$

Apply Lemma 5.3 to show that there exists $\epsilon^* > 0$ so small that for $\epsilon < \epsilon^*$

$$P\left(\widetilde{X}_t^{j,s_k,\epsilon,n}((\widetilde{l}^{k,i,n,t} + \gamma, \widetilde{r}^{k,i,n,t} - \gamma)) > 0, \text{ for some } j \neq i, i \in I_k, k = 1,\ldots K\right) < p/2.$$

Hence for any $\epsilon_1, \epsilon_2 < \epsilon^*$, we can define $\Gamma^{\epsilon_1,\epsilon_2} \subset \Omega$ such that $P(\Gamma^{\epsilon_1,\epsilon_2}) \geq 1-p$ and for any $\omega \in \Gamma^{\epsilon_1,\epsilon_2}$

$$\widetilde{X}_t^{j,s_k,\epsilon_m,n}((\widetilde{l}^{k,i,n,t}+\gamma, \widetilde{r}^{k,i,n,t}-\gamma)) = 0, \;\; \forall\, j \neq i, i \in I_k\,, k = 1, \ldots K, m = 1, 2.$$

In other words for any $\omega \in \Gamma^{\epsilon_1,\epsilon_2}$, we have

$$X_t^{\epsilon_m}((\widetilde{l}^{k,i,n,t}+\gamma, \widetilde{r}^{k,i,n,t}-\gamma)) = \widetilde{X}_t^{i,s_k,\epsilon_m,n}((\widetilde{l}^{k,i,n,t}+\gamma, \widetilde{r}^{k,i,n,t}-\gamma)), \;\; \forall\, i \in I_k\,, k = 1, \ldots K, m = 1, 2.$$

Fix an arbitrary $\omega \in \Gamma^{\epsilon_1,\epsilon_2}$. Since $\psi \in D_t^{K,n}$, we obtain that

$$\psi(\widetilde{y}^1) = \psi(\widetilde{y}^2), \tag{5.9}$$

for any $\widetilde{y}^1, \widetilde{y}^2 \in \operatorname{supp}(\widetilde{Y}_t^{\epsilon_1}) \cup \operatorname{supp}(\widetilde{Y}_t^{\epsilon_2})$ such that for every $k = 1, \ldots, K$, and some $i \in I_k$, we have $\widetilde{y}^1(t), \widetilde{y}^2(t) \in (\widetilde{l}^{k,i,n,t}+\gamma, \widetilde{r}^{k,i,n,t}-\gamma)$.

Now we can represent the set $A(\gamma)$ as follows:

$$A(\gamma) = \bigcup_{l=1}^{\infty} A^l(\gamma),$$

where $A^l(\gamma)$ are open connected intervals such that $A^l(\gamma) \cap A^m(\gamma) = \emptyset$, for all $l \neq m$. Then, for any $l \geq 1$ such that $X_t(A^l(\gamma)) > 0$, and $k = 1, \ldots, K$, there exists a unique $j(l,k)$ such that

$$A^l(\gamma) \subset (\widetilde{l}^{k,j(l,k),n,t}+\gamma, \widetilde{r}^{k,j(l,k),n,t}-\gamma). \tag{5.10}$$

Fix an arbitrary $\psi \in D_t^{K,n}$. By (5.9) and (5.10) we get that ψ is constant on each of the sets

$$\{\widetilde{y} \in \operatorname{supp}(\widetilde{Y}_t^{\epsilon_1}) \cup \operatorname{supp}(\widetilde{Y}_t^{\epsilon_2}) : \;\; \widetilde{y}(t) \in A^l(\gamma)\}, \; l \geq 1,$$

Hence for any $l \geq 1$ we can define

$$\psi_l \equiv \begin{cases} \psi(\widetilde{y}), \;\; \text{for } \{\widetilde{y} \in \operatorname{supp}(\widetilde{Y}_t^{\epsilon_1}) \cup \operatorname{supp}(\widetilde{Y}_t^{\epsilon_2}) : \;\; \widetilde{y}(t) \in A^l(\gamma)\}, & \text{if } X_t(A^l(\gamma)) > 0, \\ 0, & \text{otherwise,} \end{cases}$$

and we have

$$\int \psi(\widetilde{y}) \mathbf{1}_{\{\widetilde{y}(t) \in A^l(\gamma)\}} \widetilde{Y}_t^{\epsilon_m}(d\widetilde{y}) \;\; = \;\; X_t^{\epsilon_m}(A^l(\gamma))\psi_l\,, \;\; \forall l \geq 1, \; m = 1, 2,$$

for any $\omega \in \Gamma^{\epsilon_1,\epsilon_2}$. By combining all our estimates, we obtain

$$\begin{aligned} & P\left(\left|\langle \widetilde{Y}_t^{\epsilon_1}, \psi\rangle - \langle \widetilde{Y}_t^{\epsilon_2}, \psi\rangle\right| > \eta\right) \\ \leq \;\; & P\left(\left\{\left|\sum_{l=1}^{\infty}(X_t^{\epsilon_1}(A^l(\gamma)) - X_t^{\epsilon_2}(A^l(\gamma)))\psi_l\right| > \eta/2\right\} \bigcap \Gamma^{\epsilon_1,\epsilon_2}\right) + p \\ & + P\left((X_t^{\epsilon_1}(A(\gamma)^c) + X_t^{\epsilon_2}(A(\gamma)^c)) \left\|\psi\right\|_\infty > \eta/2\right) \\ \to \;\; & p + P\left((X_t(A(\gamma)^c) > \eta/(4\left\|\psi\right\|_\infty)\right) \qquad (\text{as } \epsilon_1, \epsilon_2 \downarrow 0) \\ \leq \;\; & 2p, \end{aligned}$$

and since p was arbitrary the proof is complete.

■

Proof of Theorem 2.1 It follows immediately from Proposition 5.4 that for any $K, n \geq 1$, $t > 0$, $\psi \in D_t^{K,n}$, $\langle \widetilde{Y}_t^\epsilon, \psi \rangle$ converges in probability. By Theorem 1.1 of [1], any limit law of $\widetilde{Y}_t^\epsilon$ belongs to $C_{M_F(C)}[0, \infty)$. Hence by Lemmas 5.1 and 5.2 we see that there exists a process $\widetilde{Y} \in C_{M_F(C)}[0, \infty)$ such that

$$\widetilde{Y}^\epsilon \to \widetilde{Y}, \text{ as } \epsilon \downarrow 0,$$

in $D_{M_F(C)}[0, \infty)$, in probability, and the proof is complete.

∎

References

[1] K. Burdzy and J. F. Le Gall. Super-Brownian motion with reflecting historical paths. *Probability Theory and Related Fields* **121**, (2001) 447–491.

[2] D. Dawson and E. Perkins. Historical Processes. *Memoirs of the American Mathematical Society*, 454, 1991.

[3] A.M. Etheridge. *An Introduction to Superprocesses*. University Lecture Series, 20. American Mathematical Society, Providence, RI, 2000

[4] J. F. Le Gall. *Spatial Branching Processes, Random Snakes and Partial Differential Equations.* Birkhäuser, ETH Zürich, 1999.

[5] J. Neveu. Erasing a branching tree. *Supplement to Adv. Applied Probability*, pages 101–108, 1986.

[6] J. Neveu and J. Pitman. The branching process in a Brownian excursion. séminare de probabilités xxiii. *Lecture Notes in Mathematics*, 1372:248–257, 1989.

[7] J. Neveu and J. Pitman. Renewal property of the extrema and tree property of the excursion of a one-dimensional Brownian motion. séminare de probabilités xxiii. *Lecture Notes in Mathematics*, 1372:239–247, 1989.